A study on irregular singular algebraic connections on curves: monodromies, canonical decompositions and an example of the determinant of period integrals

Dissertation
zur Erlangung des Doktorgrades
(Dr. rer. nat.)

dem Fachbereich 6 - Mathematik und Informatik
der Universität Essen
vorgelegt von

Byungheup Jun
aus Seoul, Korea

Essen
2002

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Tag der Disputation: 29.10.2002

Contents

Introduction

In this thesis, we study some properties of irregular connections.

On a nonsingular variety X over a field of characteristic 0, a connection ∇ is a differential operator of order one from a vector bundle E_X to $E_X \otimes \Omega^1_X$. It is the generalization of the exterior differentiation d on $\mathcal{O}_X$, which defines the de Rham complex. We extend the operator to $E \otimes \Omega^i$ by Leibniz rule. If it satisfies the relation $\nabla^2 = 0$, we say ∇ is integrable and call $(E \otimes \Omega^*_X, \nabla)$ the de Rham complex of ∇.

Originally, connections were identified with homogeneous linear differential equations over an analytic space. An analytic integrable connection yields an involutive distribution on the underlying vector bundle. Applying the Frobenius theorem for the existence of the integral manifold of a distribution, we have locally in a neighborhood of p a unique solution of ∇ with the initial value v in E_p. One can extend the local solution of ∇ (i.e. the local section of the analytic sheaf $\ker \nabla$) along a path. This analytic continuation of the solution along a loop based at p in $\pi_1(X, p)$ yields an automorphism of E_p. This in turn represents $\pi_1(X, p)$ in $\mathrm{GL}(E_p)$ and gives a monodromy representation of the fundamental group. The sheaf of local solutions of ∇ is called the local system associated to ∇. On the other hand, given an integrable connection the de Rham cohomology is defined as the (hyper)cohomology of the de Rham complex. The de Rham cohomology can be paired with the homology to produce periods.

When the underlying variety X is projective and nonsingular, all the above remains the same while passing from the algebraic connection to the analytic one via GAGA.

For non-complete varieties, one cannot relate an analytic connection with the algebraic one directly. But with the regularity of the connection at infinity, which appears in basic text books of differential equations, the theorem of existence and uniqueness of an algebraic connection for a given representation of the fundamental group was given by Deligne in [**9**]. He also gave the comparison theorem that there is a natural quasi-isomorphism between the analytic and the algebraic de Rham complex in this case, so the two de Rham cohomologies are the same. The classical Poincaré-Verdier duality also holds between the de Rham cohomology and the de Rham cohomology with compact support ([**23**]).

Without the regularity condition at infinity, that is when the connection is irregular, the above properties may not be satisfied in general. In this thesis, we are mainly interested in the monodromies, canonical decompositions and periods for irregular connections. Each topic will be presented in one chapter of this thesis.

In Chapter 1, we will handle the monodromy of irregular connections on nonsingular affine varieties. The result is based on a theorem which first appeared in a letter from Deligne to Esnault ([**10**]) showing that for any rank 1 representation of the fundamental group of an affine curve there exists a connection on the trivial bundle of rank 1 with the chosen representation as its monodromy. We will directly extract from the proof the obstruction to realize a rank 1 representation of the fundamental group as a connection on the trivial bundle of rank 1. We will generalize the theorem for rank 1 representations on any affine variety and for representations of arbitrary rank over curves. This chapter is based on the result published as [**15**].

Chapter 2 is devoted to the canonical decompositions of a formal connection at its irregular singular point after the slopes. We present the slope decomposition and the Turritin-Levelt decomposition for absolute connections. For relative connections, it is presented by several authors in different situations (cf. [**3**], [**19**] and [**21**]). While those proofs are computational, here we present them in short by virtue of lemma 6.4.3 of [**3**]. All the idea of Chapter 2 is essentially contained in loc. cit. and this presentation will be published separately in [**16**].

Finally, in Chapter 3, we calculate the period determinant of an example of an irregular connection $\nabla = d + dy$ on the affine Legendre elliptic curve $\mathrm{Spec}(k[x,y]/(y^2 - x(x-1)(x-\lambda)))$. Historically, periods are defined as an integration of "algebraic differential" forms. A general definition appears in [**18**]. Some periods are produced while taking the pairing of homology with cohomology. Bloch and Esnault developed the homology for irregular connections on curves and proved that it makes a perfect pairing with de Rham cohomology([**4**]). In the last part of [**18**], the period of the pairing between the de Rham cohomology and the homology for an irregular connection is discussed and named "exponential period". We are interested in the determinant of the period matrix in this case. On $\mathbb{P}^1$, some examples were calculated by Terasoma in [**28**] through an approximation by a sequence of the periods of regular connections. We give an example in the higher genus case, which reduces to a computation on $\mathbb{P}^1$ via the projection formula.

The period determinant will be approximated using the product formula in [**27**].

Acknowledgements

First of all, I would like to thank my thesis advisor Professor Hélène Esnault for giving me the greatest chance in my mathematical life as well as for those advises on mathematics and on the life.

I am also grateful to Professor Eckart Viehweg for warm advises and for teaching me the cohomology theory in the year 1997. It does not count in my Ph.D period, nevertheless I learned how an expert sees the objects in algebraic geometry. I thank Professor Spencer Bloch for showing interest on this work and the encouragement. Additionally I thank him for the arrangements when I visited the University of Chicago twice.

I owe Prof. Pierre Deligne the first theorem in Chapter 1 in this thesis and am thankful for reading the chapter and for some suggestions as the referee when I submitted the part. I thank Professor Bernard Malgrange for giving useful comments on the materials in Chapter 2 and for reading and accepting [**16**]. I also thank Professor Tomohide Terasoma for his mathematical work. Even though I haven't met him in person, his work inspired me to produce the result in the last chapter.

I would like to thank Silke Lekaus, Alexander Schwarzhaupt, Pedro Luis del Angel, Herbert Kanarek, Andreas Knutsen, Stefan Müller-Stach, Jaya Iyer, Tomasz Szemberg, Marco Hien, Keiji Oguiso, Jong-Won Lee, Wioletta Syszdek, Stefan Kukulies, Jinhyun Park for mathematics, drinking beer together and some other fun in life. Especially, to Silke Lekaus, I am really grateful for escorting me several times to the most dreadful place(you-know-where) in Essen.

My study in Essen was supported by the DFG-Graduiertenkolleg "Mathematische und ingenieurwissenschaftliche Methoden für sichere Datenubertragung und Informationsvermittlung" at the University of Essen.

Finally, I thank my parents. Especially, my mother taught me the powers of 2 when I was a kid before entering primary school. It was the first step of my mathematical life.

CHAPTER 1

Monodromy of algebraic connections on the trivial vector bundle

Let X be a nonsingular projective algebraic variety over $\mathbb{C}$. X_{an} will denote $X(\mathbb{C})$ the set of $\mathbb{C}$-valued points in X with strong topology. Using the existence theorem of Cauchy-Kovalevska and Serre's GAGA ([**24**]), we have a version of Riemann-Hilbert correspondence: there exists a unique algebraic vector bundle with a connection for a given linear representation of the fundamental group $\pi_1(X_{an})$ and vice versa.

Let U be a nonsingular affine open set in X and $D = X - U$ be normal crossing divisors. Then the above Riemann-Hilbert correspondence is not true any more. There may exist several algebraic connections with a given monodromy. Deligne showed that there is again the Riemann-Hilbert correspondence, requiring the regularity condition([**9**]).

In this chapter, allowing irregular singularity, we are interested in a question how far one can trivialize the underlying vector bundle of the connection for a given monodromy. A stronger version of this asks if the underlying vector bundle of the connection can be trivial. In one of the letters from Deligne to Esnault [**10**], Deligne answered without the regularity condition any rank 1-representation of the fundamental group of an affine curve can be obtained as a connection on a trivial bundle(see §1). Esnault interpreted the obstruction of the realization of the rank 1-representation on a trivial bundle as the cokernel of the map

$$H^0(U, \Omega^1_U) \xrightarrow{j} H^1(U_{an}, \mathbb{C}). \tag{1}$$

The second cohomology group is the singular cohomology group of U_{an} with $\mathbb{C}$ coefficient and can be identified with the $\mathcal{C}^\infty$ de Rham cohomology $H^1_{dR,\mathcal{C}^\infty}(U_{an})$, the analytic de Rham cohomology $H^1_{dR,an}(U) = \mathbb{H}^1(U_{an}, \Omega^*_{U_{an}})$ and the algebraic de Rham cohomology $H^1_{dR,alg}(U) = \mathbb{H}^1(U, \Omega^*_U)$ after the Grothendieck's generalized de Rham theorem. Thanks to Grothendieck's de Rham theorem, the above de Rham cohomologies are isomorphic to each other and we may denote the algebraic de Rham cohomology by $H^1_{dR}(U)$ simply.

From the construction of the hypercohomology, $\mathbb{H}^1(U, \Omega_U^*)$ is generated by $H^0(U, \Omega_U^1)$ and $H^1(U, \mathcal{O}_U)$. When U is affine, the higher coherent sheaf cohomology should be trivial so that the previous map is surjective. Then Deligne uses the duality of homology and cohomology group to realize the rank 1-representation on the trivial bundle. Once we assume the surjectivity of the map $H^1(U, \mathbb{C}) \to H^1(U, \mathbb{C}^*)$, it is extended directly to rank 1 case on higher dimensional varieties(Corollary 2 and Theorem 3).

Theorem 7 extends the similar argument for representations of arbitrary rank.

THEOREM 7. Let U be a smooth affine curve over $\mathbb{C}$. For any given rank r representation ρ of the fundamental group, there exists an algebraic connection ∇ on the trivial vector bundle of rank r with the monodromy ρ.

In section 2, we recall the iterated integral à la K.T. Chen to describe the monodromy of connections on a trivial vector bundle.

The main theorem will be proved in section 3.

As an example for the realization problem, we present any unipotent representation of rank 2 can appear as a connection on the trivial bundle(see Section 4).

We concentrate on the dimension 1 case. But some results are valid even for higher dimensional varieties.

1. Rank 1 representations

Let U be a nonsingular affine curve, X be its unique completion and $D = X - U$.

THEOREM 1 (Deligne[**10**]). *For given rank 1 monodromy representation of $\pi_1(U_{an})$, for any given rank 1 monodromy representation ρ of $\pi_1(U_{an})$, there exists an algebraic connection ∇ on $\mathcal{O}_U$ with the underlying monodromy ρ.*

PROOF. From the exponential sequence, we have the following exact sequence of singular cohomology groups:

$$H^1(U_{an}, \mathbb{C}) \xrightarrow{\exp} H^1(U_{an}, \mathbb{C}^*) \longrightarrow H^2(U_{an}, \mathbb{Z}) \to H^1(U_{an}, \mathbb{C}). \tag{2}$$

We know $H^1(U_{an}, \mathbb{C}^*)$ is the group of isomorphism classes of rank 1 local systems on U_{an} and $H^2(U_{an}, \mathbb{Z})$ vanishes by dimension reason.

Via the identification(cf. [**12**])

$$H^1_{dR}(U) := \mathbb{H}^1(U, \Omega^\bullet) = H^1(U_{an}, \mathbb{C}),$$

the de Rham class of ω in $H^0(U,\Omega^1)$ yields a class $[\omega]$ in $H^1(U_{an},\mathbb{C})$, such that

$$\exp[\omega] \in H^1(U_{an},\mathbb{C}^*) = \mathrm{Hom}(\pi_1(U_{an}),\mathbb{C}^*)$$

is the underlying monodromy of $(\mathcal{O}, d-\omega)$.

On an affine variety, the higher coherent sheaf cohomology vanishes. Thus $H^0(U,\Omega^1)$ surjects onto $\mathrm{Hom}(\pi_1(U_{an}),\mathbb{C}^*)$.

This finishes the proof. □

A clever observation of the proof makes us generalize in the following:

COROLLARY 2. *Let U be an affine variety over $\mathbb{C}$. Then the followings are equivalent.*

(1) *$H^2(U_{an},\mathbb{Z})$ is torsion-free.*
(2) *For any given rank 1 representation ρ of $\pi_1(U_{an})$, there exists an integrable algebraic connection ∇ on $\mathcal{O}_U$ with the monodromy ρ.*
(3) *If a line bundle $\mathcal{L}$ has vanishing Chern class in $H^2_{dR}(U)$, then for any rank 1 representation ρ of $\pi_1(U_{an})$, there exists an integrable algebraic connection ∇ on $\mathcal{L}$ with the monodromy ρ.*

PROOF. (1) $\Leftrightarrow$ (2) As in the proof of Theorem 1, one has the exact sequence arising from the exponential sequence (2). On an affine variety U, $\Gamma(U,\Omega^1_{closed})$ generates $H^1_{dR}(U) = H^1(U_{an},\mathbb{C})$. The the map exp is surjective if and only if

$$\Gamma(U,\Omega^1_{closed}) \to H^1(U_{an},\mathbb{C}^*)$$

is surjective of equivalently, $H^2(U_{an},\mathbb{Z})$ is torsion-free.

(2) $\Rightarrow$ (3) Consider the complex of sheaves on the Zariski topology of U:

$$\mathcal{C}^\bullet : \mathcal{O}^* \overset{d\log}{\to} \Omega^1 \overset{d}{\to} \Omega_2 \to \cdots .$$

The first hypercohomology of $\mathcal{C}^\bullet$ is the group of isomorphism classes of line bundles with integrable connection, which will be denoted by $\mathrm{Pic}^\nabla(U)$. The filtration of $\mathcal{C}$ by its degree gives

$$\begin{array}{ccccc} \mathbb{H}^1(U,\mathcal{C}^\bullet) & \longrightarrow & H^1(U,\mathcal{O}^*) & \longrightarrow & \mathbb{H}^2(U,\Omega^{\geq 1}) \\ \| & & \| & & \| \\ \mathrm{Pic}^\nabla(U) & \longrightarrow & \mathrm{Pic}(U) & \xrightarrow{c} & H^2_{dR}(U), \end{array}$$

where c denotes the first Chern class map and the last vertical map is an isomorphism since the kernel and the cokernel are

$$H^1(U,\mathcal{O}) = H^2(U,\mathcal{O}) = 0.$$

Since $c(\mathcal{L}) = 0$, from the exact sequence, there exists an integrable connection ∇_0 on $\mathcal{L}$. Let ρ_0 be the monodromy of $(\mathcal{L}, \nabla_0)$. As one can realize any given monodromy on $\mathcal{O}$, one has an algebraic connection ∇_1 with the monodromy $\rho_0^{-1} \circ \rho$ on $\mathcal{O}$. Twisting $(\mathcal{L}, \nabla_0)$ with $(\mathcal{O}, \nabla_1)$, one has the connection $\nabla = \nabla_0 \otimes \nabla_1$ on $\mathcal{L}$ with the desired monodromy ρ.

(3) $\Rightarrow$ (2) Clear. □

As a special case of the corollary, we have the following:

Theorem 3. *Let X be a smooth projective variety over $\mathbb{C}$ and D be a normal crossing divisor such that the complement U is affine and such that the components generate the Néron-Severi group $NS(X)$ of X. Then for any rank 1 monodromy ρ in $\mathrm{Hom}(\pi(U_{an}), \mathbb{C}^*)$, there exits an algebraic connection $\nabla = d - \omega$ on $\mathcal{O}_U$ with the underlying monodromy ρ.*

Proof. By the equivalence of Corollary 2, it suffices to show that $H^2(U_{an}, \mathbb{Z})$ is torsion-free. But in the exact sequence (2), torsion comes from $H^1(U_{am}, \mathbb{C}^*)$. Using the existence theorem in [**9**], one has a line bundle on X with connection $(\mathcal{L}, \nabla)$ which is regular singular at D with the underlying monodromy ρ. The image of $\mathcal{L}$ maps into $NS(X)$, and thus a torsion element in $H^2(U_{an}, \mathbb{Z})$ vanishes. Therefore $H^2(U_{an}, \mathbb{Z})$ is torsion-free. □

Remark 1. With a commutative diagram, one can summarize the result in a simple form where all rows and columns are exact:

(3)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{H^0(U,\Omega^1)_{\mathbb{Z}}}{d\log H^0(U,\mathcal{O}^*)} & \to & \frac{H^0(U,\Omega^1_{closed})}{d\log H^0(U,\mathcal{O}^*)} & \to & H^1(U_{an}, \mathbb{C}^*) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Pic}^{\nabla_0}(U) & \to & \mathrm{Pic}^{\nabla}(U) & \to & \mathrm{Hom}(\pi_1(U_{an}), \mathbb{C}^*) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Pic}(U) & \to & \mathrm{Pic}(U) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^2_{dR}(U) & = & H^2_{dR}(U) & & & &
\end{array}$$

In the above diagram, $\mathrm{Pic}^{\nabla_0}(U)$ is the subgroup of $\mathrm{Pic}^{\nabla}(U)$ with respect to the trivial monodromy and $H^0(U, \Omega^1)_{\mathbb{Z}}$ is the group of global 1-forms with integral periods. When $\dim U = 1$, one has $H^2_{dR}(U) = 0$. Thus, we have $\mathrm{Pic}^{\nabla}(U)$ and $\mathrm{Pic}^{\nabla_0}(U)$ as extensions of $\mathrm{Pic}(U)$.

2. Chen's iterated integral

Suppose $W_1, \ldots, W_k$ are matrices in $M(r \times r, \Gamma(X, \Omega^1))$ and $\gamma : [0,1] \to X$ a piecewise smooth path. Then the iterated integral of $W_1, \ldots, W_k$ over γ is defined as

$$\int_{\Delta_k \gamma} W_1 \ldots W_k = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-1}} w_1(t_1) w_2(t_2) \cdots w_k(t_{k-1}) dt_k \cdots dt_1, \tag{4}$$

where $w_i = \gamma^* W_i$.

Let $\nabla = d - A$ be a connection on a trivial bundle of rank r. Thus $A \in \Gamma(X, \Omega^1 \otimes \mathfrak{gl}(r))$.

PROPOSITION 4 (Chen[**7**],[**8**]). *Suppose γ is a loop, then the monodromy of the connection $\nabla = d - A$ along γ is given by the absolute convergent series of the iterated integrals.*

$$m(\gamma) = I + \int_\gamma A + \int_{\Delta_2 \gamma} AA + \int_{\Delta_3 \gamma} AAA + \cdots \in \mathrm{GL}(r, \mathbb{C}). \tag{5}$$

PROOF. (compare to [**7**], [**8**] and [**25**]) Let us prove the absolute convergency of the series first. The k-th term of the series is bounded by

$$\begin{aligned} || \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \underbrace{A \cdots A}_{k\text{-times}} || &\leq \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} ||A||^k \\ &\leq \frac{1}{k!} |\gamma|^k ||A||^k, \end{aligned} \tag{6}$$

where $|\gamma|$ is the length of the loop γ and $||A||$ is the matrix norm of A. Thus the summation is bounded by the $\exp |\gamma| \|A\|$. Hence the series is absolute convergent.

Let us denote the k-th term of the summation by $I_k(\gamma(t))$. Then we have

(7)
$$\begin{aligned}
\nabla_{\gamma'(t)} I_k(\gamma(t)) &= (d/dt - A(t))I_k(\gamma(t)) \\
&= \frac{d}{dt}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} A(t_1)A(t_2)\ldots A(t_k)dt_k\ldots dt_2dt_1 \\
&\quad - A(t)I_k(\gamma(t)) \\
&= \int_0^t\int_0^{t_2}\cdots\int_0^{t_{k-1}} A(t)A(t_2)\ldots A(t_k)dt_k\ldots dt_2 \\
&\quad - A(t)I_k(\gamma(t)) \\
&= A(t)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-2}} A(t_1)A(t_2)\ldots A(t_{k-1})dt_{k-1}\ldots dt_2dt_1 \\
&\quad - A(t)I_k(\gamma(t)) \\
&= A(t)I_{k-1}(\gamma(t)) - A(t)I_k(\gamma(t)).
\end{aligned}$$

Now we can calculate the whole covariant derivative of $m(\gamma)$ along γ:

(8)
$$\begin{aligned}
\nabla_{\gamma(t)} m(\gamma) &= \nabla_{\gamma'(t)}[I + \int_\gamma A + I_2(\gamma(t)) + I_3(\gamma(t)) + \ldots] \\
&= -A(t) + [A(t) - A(t)\int_\gamma A] + [I_1(\gamma(t)) - A(t)I_2(\gamma(t))] \\
&\quad + [A(t)I_3(\gamma(t)) - A(t)I_4(\gamma(t))] + \ldots \\
&= [-A(t) + A(t)] + [-A(t)I_1(\gamma(t)) + A(t)I_1(\gamma(t))] \\
&\quad + [-A(t)I_2(\gamma(t)) + A(t)I_2(\gamma(t))] + \ldots \\
&= 0.
\end{aligned}$$

Therefore $m(\gamma)$ gives the monodromy around the loop γ. □

Corollary 5 (Chen[**7**]). *On the structure sheaf $\mathcal{O}$ with connection $\nabla = d - \omega$ where ω is an 1-form on the affine open set of a smooth curve, the monodromy along γ is given by*

$$m(\gamma) = \exp\int_\gamma \omega. \tag{9}$$

Proof. (compare to [**7**]) When the bundle is trivial of rank 1, the iterated integral can be seen as the Lebesgue integral of

$$\begin{aligned}
A(t_1, ..., t_k) &\stackrel{def}{=} W(t_1)W(t_2)\cdots W(t_k) \\
&= \gamma^*\omega(t_1)\gamma^*\omega(t_2)\cdots\gamma^*\omega(t_k)
\end{aligned} \tag{10}$$

over $\Delta_k = \{(t_1, t_2, \ldots, t_k) \in [0,1]^k | 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq 1\}$. The integration gives a complex measure on $[0,1]^k$. Let $\mu(M)$ be $\int_M A$ for a measurable set M in $[0,1]^k$.

We have

$$\begin{aligned} \mu([0,1]^k) &= \int_0^1 \int_0^1 \cdots \int_0^1 W(t_1)W(t_2)\cdots W(t_k)dt_k \ldots dt_1 \\ &= [\int_0^1 W(t)dt]^k = [\int_\gamma \omega]^k \end{aligned} \tag{11}$$

by Fubini theorem.

On the other hand, we have again

$$\begin{aligned} \mu([0,1]^k) &= \sum_{\sigma \in S_k} \mu(\Delta_k^\sigma) \\ &= k!\mu(\Delta_k) \end{aligned} \tag{12}$$

where S_k is the permutation group of k-elements and Δ_k^σ is the following k-simplex

$$\Delta_k^\sigma = \{(t_1, ..., t_k) \in I^k | \sigma(t_1) \leq \ldots \leq \sigma(t_k)\}. \tag{13}$$

It follows that

$$\begin{aligned} I_k(\gamma) &= \mu(\Delta_k) \\ &= \frac{1}{k!}\mu(I^k) = \frac{1}{k!}[\int_\gamma \omega]^k. \end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned} m(\gamma) &= \sum_{k=0}^\infty I_k(\gamma) = \sum_{k=0}^\infty \frac{1}{k!}[\int_\gamma \omega]^k \\ &= \exp \int_\gamma \omega. \end{aligned} \tag{15}$$

□

Remark 2. Compared with the local monodromy formula in [**9**], the previous corollary gives the same result from another side because $\mathrm{Res}_p(\nabla)$ is $(-1/2\pi i)\int_\gamma \omega$ in this case, where γ is a simple loop around p.

3. Realization of monodromy representations of arbitrary rank

This section begins with a lemma due to Serre. We recall its geometric version due to Atiyah[**1**].

LEMMA 6 (Serre). *Let E be a vector bundle of rank r on an affine curve U, then*

$$E = \mathcal{O}^{r-1} \oplus \mathcal{L}, \tag{16}$$

where $\mathcal{L} = \det E = c_1(E) \in \operatorname{Pic}(U)$.

PROOF. (Atiyah [**1**]) If $\operatorname{rank}(E) = 1$, then $\det(E) = E$. So we have nothing to prove.

Suppose that there exists a global nonvanishing section s in the vector bundle E whose rank is greater than 1. Then we have a vector bundle $E' = E/\mathcal{O}$ defined by the exact sequence:

$$\begin{array}{c} 0 \to \mathcal{O} \to E \to E' \to 0. \\ 1 \mapsto s \end{array} \tag{17}$$

The above exact sequence is splitting, since $\operatorname{Ext}^1(E', \mathcal{O})$ is trivial on the affine curve. Hence we have $E = \mathcal{O} \oplus E'$. If E' is of rank 1, then there remains nothing to do. When E' is of rank greater than 1, then the existence of global nonvanishing section in the vector bundle on affine curves concludes the proof.

Let X be the smooth projective completion of U and E be a vector bundle of rank r on U. And denote a locally free sheaf F on X such that $F|_U = E$. Since the divisor $D = X - U$ is ample in X, $F(\ell \cdot D)$ is globally generated for a sufficiently large ℓ on X and we have still $F(\ell \cdot D)|_U = E$. Hence there is an exact sequence of sheaves:

$$0 \to K \xrightarrow{i} \mathcal{O}^{r+k} \xrightarrow{p} E \to 0, \tag{18}$$

where K is the kernel of p.

Tensoring the previous exact sequence with $\kappa(x) = \mathcal{O}_x/m_x$ for $x \in U$, we have the following exact sequence of vector spaces.

$$0 \to K_x \xrightarrow{i_x} \mathbb{C}^{r+k} \to E_x \to 0. \tag{19}$$

The first inclusion yields the inclusion of projective spaces:

$$\mathbb{P}(K_x) \xrightarrow{\mathbb{P}_x(i)} \mathbb{P}^{r+k-1}. \tag{20}$$

On the whole U, we have the inclusion of the associated projective bundles

$$\mathbb{P}(K) \xrightarrow{\mathbb{P}(i)} \mathbb{P}(\mathcal{O}^{r+k}) = \mathbb{P}^{r+k-1} \times U \xrightarrow{\pi_1} \mathbb{P}^{r+k-1}, \tag{21}$$

where π_1 is the first projection.

Then we have, since $r = \operatorname{rank}(E)$ is greater than 1,

$$\begin{aligned} \dim(\pi_1 \circ \mathbb{P}(i)(\mathbb{P}(K))) &\le \dim(\mathbb{P}(K_x)) + \dim(U) \\ &= k < r + k - 1 = \dim(\mathbb{P}^{r+k-1}). \end{aligned} \tag{22}$$

Hence the map cannot be onto and we have a point p outside of the image of the $\pi_1 \circ \mathbb{P}(i)$ in $\mathbb{P}^{r+k-1}$. It corresponds to a global section of E nowhere vanishing on U. $\square$

REMARK 3. The previous theorem presented above holds for any projective module M of finite rank r over a Dedekind domain A. We have $M \cong A^{r-1} \oplus L$, where $L \in \mathrm{Pic}(A)$. The above theorem is a special case when $U = \operatorname{Spec} A$.

As a direct consequence, a vector bundle E on a nonsingular affine curve is trivial if and only if $\det E = \mathcal{O}$.

Finally we present the main theorem of this chapter.

THEOREM 7. *Let U be a smooth affine curve over $\mathbb{C}$. For any given rank r representation ρ of the fundamental group, there exists an algebraic connection ∇ on the trivial vector bundle of rank r with the monodromy ρ.*

PROOF. Let $X = U \cup D$ be the smooth completion of U. Using the existence theorem in [**9**], there exists a vector bundle E on X with a connection ∇_0 regular singular at D, such that its restriction on U gives the monodromy ρ.

By the decomposition theorem of a vector bundle, we have $E|_U \cong \mathcal{O}^{r-1} \oplus \mathcal{L}$, where $\mathcal{L} = \det(E|_U)$. Since $\mathrm{Pic}(U)$, as a quotient of a divisible group $\mathrm{Pic}^0(X)$, is divisible, one has a line bundle $\mathcal{M} \in \mathrm{Pic}(U)$ such that $\mathcal{M}^{\otimes r} = \mathcal{L}$. Moreover, using the equivalences in the Corollary 2, $\mathcal{M}$ can be equipped with an algebraic connection ∇_1 with the trivial monodromy as $H^2_{dR}(U)$ vanishes. Twisting $(E|_U, \nabla_0)$ with $(\mathcal{M}, \nabla_1)^{\otimes -1}$ trivializes the underlying vector bundle of the connection $\nabla_0 \otimes \nabla_1^{\otimes -1}$ since

$$\det(E|_U \otimes \mathcal{M}^{-1}) = \mathcal{L} \otimes \mathcal{M}^{\otimes -r} = \mathcal{O} \quad \text{in } \mathrm{Pic}(U).$$

$\square$

4. Unipotent rank 2 representations

Now we know on a curve every monodromy representation can be realized on the trivial bundle. Deligne's construction for the realization on the trivial bundle of rank 1 can be seen as a specific example of the main theorem in the previous section. We can construct one more example after the idea of Deligne.

Let us recall the result of Deligne in the first section. It says that for any rank 1 representation of the fundamental group of the open affine curve, there exists an algebraic connection on the structure sheaf with the given underlying monodromy.

Let U be a nonsingular affine variety over $\mathbb{C}$.

Given a representation ρ of $\pi_1(U_{an})$ paths through $H_1(U_{an})$. Then there exists a closed differential form ω in $H^0(U, \Omega)$ such that

$$\rho(\gamma) = \exp \int_\gamma \omega, \tag{23}$$

for every γ in $\pi_1(U)$. It is possible because $H^1(U, \mathbb{C})$ is dual to $H_1(U_{an})$ by the Poincaré duality and $H^0(U, \Omega)$ surjects to the first de Rham cohomology group.

Now the same idea can be applied to the unipotent rank 2 case. There is an isomorphism between $\mathbb{G}_a$ and $U(2)$ the unipotent matrix group of rank 2. The isomorphism is given as following:

$$\begin{aligned} \mathbb{G}_a &\xrightarrow{\simeq} U(2) \\ a &\mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{24}$$

Then any representation ρ can be written as

$$\rho(\gamma) = \begin{pmatrix} 1 & \tilde{\rho} \\ 0 & 1 \end{pmatrix} \tag{25}$$

where $\tilde{\rho}$ is an additive representation of the fundamental group to $\mathbb{G}_a$.

The following theorem is, in fact, a consequence of Corollary 2. But it shows the shape of a connection realizing unipotent rank 2 representation.

THEOREM 8. *For any unipotent rank* 2 *representation, there exists an algebraic connection* ∇ *on the rank* 2 *trivial bundle whose underlying local system corresponds to the prescribed monodromy representation* ρ *if and only if any rank* 1 *representation can be found as the underlying monodromy of a connection on the trivial bundle.*

PROOF. From the above paragraph, it suffices to show that there exists a closed differential form ω such that $\tilde{\rho}(\gamma) = \int_\gamma \omega$ by the same idea as Deligne on the rank 1 case. We can see the representation can be realized on the trivial bundle by the connection $\nabla = d - A$ where the connection matrix is

$$A = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}. \tag{26}$$

The monodromy can be calculated by Chen's iterated integral. The monodromy of $\nabla = d - A$ around γ is:

$$
\begin{aligned}
I + &\int_\gamma A + \int_{\Delta_2\gamma} AA + \int_{\Delta_3\gamma} AAA \dots \\
&= I + \int_\gamma A \quad \text{(for } A^2 = 0.) \\
&= I + \begin{pmatrix} 0 & \int_\gamma \omega \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \tilde{\rho}(\gamma) \\ 0 & 1 \end{pmatrix} \\
&= \rho(\gamma),
\end{aligned} \tag{27}
$$

which fulfills the desired monodromy. $\square$

CHAPTER 2

Canonical decomposition of a formal connection at an irregular singular point

In this chapter we recall Malgrange's theorem 3.2.1 in [**21**] and give a short and simple argument for the proof: If ∇ is an absolute integrable connection on $R((z)) := \mathbb{C}\{x_1, x_2, \ldots, x_n\}((y))$, then the Levelt-Turritin decomposition of the R-relative connection ∇_{rel} after ramification $y^{1/N}$ is stabilized by ∇. Indeed Malgrange's theorem is reformulated by André and Baldassarri in [**2**] algebraically. Applying Galois descent for $R((y^{1/N}))$ over $R((y))$, we have a corollary of the result that the slop decomposition on $R((y))$ of ∇_{rel} is also stabilized by ∇.

Here, in the sprit of Lemma 6.4.3 of [**3**], while both proofs are computational, we don't need any explicit computation on the order of poles. Firstly, we observe the slope decomposition in one variable is stabilized under ∇. Then starting from the slope decomposition, passing by a standard argument (splitting lemma), we conclude that the Turritin-Levelt decomposition after ramification is also absolute.

The material presented in this chapter was known already in [**2**], but as in Chap.3 of [**20**], starting from the fact that the slope decomposition stabilized, we can conclude that the classical (but complicated) Turritin-Levelt decomposition is stabilized by ∇.

1. Slope decomposition of a relative connection

Let k be a field of characteristic 0 and K be any extension of k. Throughout this chapter, the term "absolute" means relative to k.

Denote by $\mathcal{D}$ the ring of differential operator over $K((z))$. An absolute connection ∇ on V a finite dimensional vector space over $K((z))$ is a differential operator of order 1 from V with values in $V \otimes_k \Omega_{K((z))/k}$

$$\nabla : V \to V \otimes_k \Omega_{K((z))/k}, \tag{28}$$

satisfying the Leibniz rule $\nabla(f \cdot v) = df \cdot v + f\nabla(v)$ for $f \in K((z))$ and $v \in V$.

It is said to be integrable if $\nabla^2 = 0$. One associates a unique connection ∇_{rel} to ∇. ∇_{rel} endows V with a natural $\mathcal{D}$-module structure.

Let the rank of V be r. A vector v in V is called cyclic vector if $\{v, Dv, \dots, D^{r-1}v\}$ is a basis of V for $D := d/dz$. It is well-known that V has a cyclic basis (cf. [**9**], Chap.2 and [**20**], Chap.3).

A cyclic basis satisfies a polynomial in D:

$$av := (D^n + a_{n-1}D^{n-1} + \dots + a_0)v = 0, \tag{29}$$

for some $a_0, \dots, a_{n-1}$ in $K((z))$.

When v is a cyclic basis satisfying the polynomial a,

$$V \cong \mathcal{D}/\mathcal{D}_a. \tag{30}$$

Definition 1. For a nonzero differential operator

$$a := D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0,$$

the Newton polygon of a is the convex hull of the quadrants $\{u \leq k, v \geq k - v(a_k) | v(a_k) < \infty\}$ in $\mathbb{R}^2$. We call the slopes of the non-vertical edges of the Newton polygon of a in the right half plane, the slopes of a. If $V \cong \mathcal{D}/\mathcal{D}a$, we call the Newton polygon of a the Newton polygon of V and the slopes of a the slopes of V.

The definitions of the Newton polygon and the slopes for V are well-defined, since they are independent of the choice of a cyclic vector in V(cf. Chap.3 in [**20**]).

Let $0 \leq p_1 < p_2 < \dots < p_s$ be the slopes of $V = \mathcal{D}/\mathcal{D}a$. Then there exists b, c in $\mathcal{D}$ such that $a = bc$, where the slopes of b (resp. c) are $p_1, \dots, p_r$ (resp. $p_{r+1}, \dots, p_s$) for a integer between $1, s$. It yields a short exact sequence of $\mathcal{D}$-modules:

$$0 \to \frac{\mathcal{D}}{\mathcal{D}b} \to \frac{\mathcal{D}}{\mathcal{D}a} \to \frac{\mathcal{D}}{\mathcal{D}c} \to 0.$$

Moreover, the above short exact sequence splits in a unique manner.

Lemma 9 ([**20**], 3.1.4). *Suppose V and W have different slopes, then*

$$\mathrm{Ext}^i_{\mathcal{D}}(V, W) = \mathrm{Ext}^i_{\mathcal{D}}(V, W) = 0,$$

for $i = 0, 1$.

A direct consequence of the previous lemma is that we have the unique slope decomposition of V:

$$V = \bigoplus_i V_i,$$

where V_i has only a single slope.

2. Slope decomposition of an absolute connection

In this section, we introduce the lemma 6.4.3 of [**3**], which plays the bridge from the relative to the absolute case. We still keep the same conventions and notations as before and any connection in this section is assumed integrable.

LEMMA 10 ([**3**]). *Let* (V, ∇) *be an absolute connection on* $K((z))$. *If*

$$(V, \nabla_{rel}) = (V_1, \nabla_{rel,1}) \oplus (V_2, \nabla_{rel,2})$$

and

$$\mathrm{Hom}_{\mathcal{D}}(V_1, V_2) = 0,$$

then the absolute connection ∇ *stabilizes both of* V_1, V_2 *(i.e.* $\nabla(V_i) \subset V_i \otimes \Omega^1_{K((z))/k}$*). Thus*

$$(V, \nabla) = (V_1, \nabla_1) \oplus (V_2, \nabla_2),$$

where ∇_i *is an absolute connection on* V_i *whose associated relative connection is* ∇_{rel}.

PROOF. (compare with [**3**], 6.4.3) ∇ stabilizes V_i if and only if the $K((z))$-linear map

$$\begin{aligned} \eta_{ij} : V_i &\longrightarrow V_j \otimes \Omega_{K//k} \\ v &\mapsto \nabla(v) \qquad \mod V_i \otimes \Omega_{K/k} \end{aligned} \tag{31}$$

is zero for $i \neq j$.

$K((z)) \otimes \Omega_{K/k}$ has a natural $\mathcal{D}$-module structure

$$\frac{d}{dz} : f \otimes \omega \mapsto \frac{df}{dz} \otimes \omega$$

as a subspace of $\Omega_{K((z))/k}$.

Hence $V_j \otimes \Omega_{K/k}$ is equipped with a natural $\mathcal{D}$-module structure which agrees with the action of ∇_j on $V_j \otimes \Omega_{K((z))/k}$. The integrability of ∇ yields

$$\eta_{ij} \circ \nabla_{i,d/dz} = \nabla_{j,d/dz} \circ \eta_{ij}.$$

Therefore we obtain

$$\eta_{ij} \in \mathrm{Hom}_{\mathcal{D}}(V_i, V_j \otimes \Omega_{K/k}) = 0.$$

□

Together with lemma 10, the previous lemma yields the decomposition theorem for absolute connections as in [**21**]:

THEOREM 11. *Let (V, ∇) be an absolute connection on $K((z))$. Then there exists a unique decomposition of V*

$$(V, \nabla) = \bigoplus_i (V_i, \nabla_i),$$

where V_i has a unique slope p_i as a $\mathcal{D}$-module.

3. Turritin-Levelt decomposition of an absolute connection

Recall the well-known decomposition theorem of a connection due to Turritin([**29**]), Levelt([**19**]). It is a direct sum decomposition of a connection on $K((z))$ with each direct summand of the form:

$$(L \otimes M, \nabla_L \otimes \nabla_M),$$

where
1. (L, ∇_L) is a connection of rank 1 over $K((z))$, and
2. (M, ∇_M) is regular singular at (z).

Let $(L_1 \otimes M_1, \nabla_{L_1} \otimes \nabla_{M_1})$ and $(L_2 \otimes M_2, \nabla_{L_2} \otimes \nabla_{M_2})$ be two direct summands of Turritin-Levelt decomposition such that L_1, L_2 are not isomorphic to each other.

Then it follows

$$\mathrm{Ext}^i_{\mathcal{D}}(L_1 \otimes M_1, L_2 \otimes M_2) = \mathrm{Ext}^i_{\mathcal{D}}(M_1, L_1^{-1} \otimes L_2 \otimes M_2) = 0$$

for $i = 0, 1$.

Recall the main theorem of [**19**]:

THEOREM 12 (Levelt [**19**]). *Let (V, ∇) be a connection over $K((z))$ relative to K. Then (V, ∇) has a unique Turritin-Levelt decomposition after finite extension $\hat{K}$ of K and a ramification $z = t^n$. (i.e. $(V \otimes \hat{K}((t)), \nabla \otimes d)$ has Turritin-Levelt decomposition.)*

This is the relative version of the theorem, which we now extend it to the absolute case.

Returning to the absolute case, let (V, ∇) be a absolute integrable connection over $K((z))$. Applying Lemma 9, 10, we obtain the following from the above theorem.

THEOREM 13. *Let (V, ∇) be an absolute integrable connection over $K((z))$. Then the Turritin-Levelt decomposition of its relative connection obtained after finite extension $\hat{K}$ of K and finite ramification $z = t^n$, is stabilized by ∇. Hence one has a unique Turritin-Levelt decomposition of an absolute connection*

$$(V, \nabla) = \bigoplus_i (L_i, \nabla_{L_i}) \otimes (M_i, \nabla_{M_i}),$$

where $L_i \neq L_j$ *for* $i \neq j$.

CHAPTER 3

Period of an irregular connection over an elliptic curve: $\nabla = d + dy$

In this chapter, we will calculate the period determinant of the connection $\nabla = d + dy$ over the Legendre elliptic curve: $y^2 = x(x-1)(x-\lambda)$. ∇ has an irregular singularity at ∞. The period for irregular connections is defined to be a pairing of de Rham cohomology and a newly defined homology in [**4**]. It is shown to be perfect.

For a regular singular connection, the period is defined as a pairing of de Rham cohomology and the homology relative to the singularities valued in its dual local system, via a singular integration. From the view of the theory of characteristic classes, it was discussed profoundly by T. Saito and T. Terasoma in [**23**]. Especially on $\mathbb{P}^1$, it is described by Terasoma explicitly as a product formula written in terms of the Gamma factors and the tame symbols in [**27**].

As for the irregular connections, very few cases are known. On $\mathbb{A}^1 - \{\lambda_1, \ldots, \lambda_n\}$, when the connection is of the form $d + \sum \frac{s_i}{x-\lambda_i} + df$ for a polynomial f and positive real numbers s_i , the period determinant is calculated via approximation by series of rank 1 regular singular period ([**28**]).

In our case, the period determinant involves some elliptic integrals, which are not easily calculated. Let π be the second projection of U onto $\mathbb{A}^1$, $(x, y) \mapsto y$. Then the connection we consider is pulled-back from $\mathbb{P}^1$ via π^*. Therefore we apply the projection formula to reduce the period computation on the elliptic curve to a computation on $\mathbb{P}^1$ where we then mostly apply Terasoma's idea.

Let $D \subset \mathbb{P}^1$ be those ramification points and d_D be the standard exterior differentiation of $\mathcal{O}(-D)$ in $\mathcal{O}$. If necessary, tensored with d_D, $\pi_*(\nabla)$ will be approximated by connections whose periods are computable using the product formula. Returning to the original connection, we take another connection $\nabla_\Sigma = \nabla \otimes d_\Sigma$, where $\Sigma := \pi^{-1}D$. This new connection on U produces the same periods as $d_D \otimes \pi_*(\nabla)$ since they have the same relative homology groups as well as the same de Rham cohomology groups. Finally, we will compare the periods of

∇ with that of ∇_Σ through the long exact sequence of de Rham cohomology and of the homology to produce exact value of the period determinant of ∇ on U.

1. De Rham cohomology and homology for irregular singular connection

In this section, we will calculate the de Rham cohomology and the homology defined in [**4**], for the irregular connection $d + dy$ on X a complete Legendre curve in $\mathbb{P}^2$. This has an irregular singular point of irregularity 3 at infinity and its monodromy is trivial.

1.1. de Rham cohomology. In general, for an integrable connection $\nabla : E \to E \otimes \Omega^1_U$ on a nonsingular variety U over a field of characteristic 0, the de Rham cohomology of ∇, $H^i_{dR}(U, \nabla)$ is defined as the hypercohomology of the associated de Rham complex.

In our case, since $U = X - \{\infty\}$ is an affine curve and $E = \mathcal{O}$, $H^i_{dR}(U, \nabla) = 0$ for $i > 1$ and

$$\begin{aligned} H^1_{dR}(U, \nabla) :=& \mathbb{H}^1(U, (\Omega^*_U, \nabla)) = \frac{\Gamma(U, \Omega^1)}{\nabla\Gamma(U, \mathcal{O})} \\ =& \frac{\Gamma(X, \Omega^1(*\infty)}{\nabla\Gamma(X, \mathcal{O}(*\infty))}. \end{aligned}$$

We have a natural inclusion

$$\begin{array}{ccc} \mathcal{O}(n \cdot \infty) & \xrightarrow{\nabla} & \Omega((4+n) \cdot \infty) \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\ \mathcal{O}(*\infty) & \xrightarrow{\nabla} & \Omega(*\infty), \end{array} \tag{32}$$

whose cokernel is

$$\bigoplus_{l>n} k < \frac{1}{t^l} >^{\nabla} \xrightarrow{\text{mod } \frac{1}{t^{n+4}}} \bigoplus_{l>n+4} k < \frac{dt}{t^l} > .$$

Since $\nabla \mod \frac{1}{t^{n+4}}$ is an isomorphism of k-vector space, Coker i is quasi-isomorphic to 0.

Thus we have obtained the de Rham cohomology of ∇ in another way

$$H^1_{dR}(U, \nabla) = \mathbb{H}^1(X, \mathcal{O}(n \cdot \infty) \xrightarrow{\nabla} \Omega((4+n) \cdot \infty)),$$

for any n.

Taking the filtration on the complex

$$\mathcal{O}(n \cdot \infty) \xrightarrow{\nabla} \Omega((n+4) \cdot \infty)$$

by its degree, we have the following short exact sequence:

$$
\begin{array}{ccc}
0 & \longrightarrow & \Omega((4+n)\infty) \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}(n\infty) & \xrightarrow{\nabla} & \Omega((4+n)\infty) \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}(n\infty) & \longrightarrow & 0
\end{array}
\tag{33}
$$

This yields again the long exact sequence of cohomologies:

(34)

$$
\begin{aligned}
\ldots \to \Gamma(X, \mathcal{O}(n \cdot \infty)) \xrightarrow{\nabla} \Gamma(X, \Omega((4+n) \cdot \infty)) \xrightarrow{\alpha} H^1_{dR}(U, \nabla) \\
\to H^1(X, \mathcal{O}(n \cdot \infty)) \xrightarrow{\beta} H^1(X, \Omega((n+4) \cdot \infty)) \to \ldots
\end{aligned}
$$

We want to know when the map α is surjective. α is surjective if and only if β is injective. When $n = 0$, since $h^1(\mathcal{O}(0 \cdot \infty)) = h^0(\mathcal{O}) = 1$ and $h^1(\Omega(4 \cdot \infty)) = h^0(\mathcal{O}(-4 \cdot \infty)) = 0$, β can not be injective. For $n > 0$, $h^1(\mathcal{O}(n \cdot \infty) = h^0(-n \cdot \infty) = 0$, so β is always injective.

It follows that

$$\Gamma(X, \Omega((n+4) \cdot \infty)) \xrightarrow{\alpha} H^1_{dR}(U, \nabla)$$

is surjective for $n > 0$.

As a consequence, when $n = 1$, we have

Proposition 14.

$$H^1_{dR}(U, \nabla) = \frac{\Gamma(X, \Omega_X(5 \cdot \infty))}{\nabla \Gamma(X, \mathcal{O}_X(\infty))}.$$

Therefore, $H^1_{dR}(U, \nabla)$ is generated by $\Gamma(X, \Omega(5 \cdot \infty))$ with the relation

$$
\begin{aligned}
\nabla \Gamma(X, \mathcal{O}(\infty)) =& \nabla \Gamma(X, \mathcal{O}) \\
=& \textit{the rank } 1 \textit{ vector space generated by } dy.
\end{aligned}
$$

So

$$\frac{dx}{y}, x\frac{dx}{y}, dx, xdx.$$

make a basis of $H^1_{dR}(U, \nabla)$.

1.2. Homology. We denote by H_i^{irreg} the homology for irregular connections defined by Bloch and Esnault in [**4**]. This homology group is defined for irregular singular connections over curves. It is defined for $i = 0, 1, 2$. Note that $H_2^{irreg}(U, \nabla^*)$ vanishes in this case for dimension reason (cf. [**4**]). This makes a perfect pairing with the de Rham cohomology of its dual connection. It is composed of topological cycles and chains on which the solution of the connection decays rapidly.

Since the pairing of de Rham cohomology of a connection and the homology of the dual connection is perfect, one concludes from the previous section that

$$\operatorname{rank} H_1^{irreg}(U, \nabla^*) = 4, \qquad \text{and,} \tag{35}$$

$$\operatorname{rank} H_0^{irreg}(U, \nabla^*) = 0. \tag{36}$$

We will describe the cycles generating $H_1^{irreg}(U, \nabla^*)$ here.

Let ∇^* be the dual of ∇, Δ be a small disk around ∞ and $\Delta^* = \Delta - \{\infty\}$.

Then Theorem 1.1 in [**4**] tells us that there is a homology long exact sequence of the form :

$$\begin{aligned} 0 \longrightarrow H_1(U, \mathbb{C}) \overset{1\otimes\exp y}{\longrightarrow} H_1^{irreg}(U, \nabla^*) &\longrightarrow H_1^{irreg}(\Delta^*, \partial\Delta, \nabla^*) \\ \overset{\delta}{\longrightarrow} H_0(U, \mathbb{C}) &\longrightarrow H_0^{irreg}(U, \nabla^*) \to 0. \end{aligned} \tag{37}$$

In the above sequence, $H_1(U, \mathbb{C}) = H_1(X, \mathbb{C})$ which is generated by γ_1, γ_2. Therefore we have still two other cycles that map to the kernel of

$$\delta : H_1^{irreg}(\Delta^*, \partial\Delta, \nabla^*) \to H_0(U, \mathbb{C}) = \mathbb{C} < \exp y > .$$

Let us describe the cycles in $H_1^{irreg}(\Delta^*, \partial\Delta, \nabla^*)$. Fix a local parameter t at ∞, so that we can write the solution of ∇^* locally $\exp(-t^3)$. Then on a small disk around ∞, we have sectors along which the solution decays rapidly :

$$\frac{2(i-1)\pi}{3} - \frac{\pi}{6} < \arg t < \frac{2(i-1)\pi}{3} + \frac{\pi}{6}, \qquad i = 1, 2, 3.$$

We call the above the *i-th rapid decay sector.*

Let η_i be a chain from a fixed point p at the boundary of Δ to ∞ along the i-th rapid decay sector. Since there are three such sectors, $\eta_i \otimes \exp y$, for $i = 1, 2, 3$ generate $H_1^{irreg}(\Delta^*, \partial\Delta, \nabla^*)$. Then the map δ is the augmentation

$$\sum_i a_i \eta_i \otimes \exp y \mapsto \sum_i a_i < \exp y >$$

and the kernel is generated by $(\eta_1 - \eta_2) \otimes \exp y$ and $(\eta_2 - \eta_3) \otimes \exp y$.

1.3. Period matrix. We can describe now the period matrix of the connection. From now on, k is a fixed subfield of $\mathbb{C}$ containing λ. (i.e. it is the field of definition of the elliptic curve.)

We have seen $\Gamma(X, \Omega(5\infty))$ generates $H^1_{dR}(U, \nabla)$ and

$$\frac{dx}{y}, x\frac{dx}{y}, dx, xdx$$

make a basis of the k-vector space $H^1_{dR}(U, \nabla)$. We will denote the generators by $\omega_1, \omega_2, \omega_3$ and ω_4, respectively.

In the other hand, we have the cycles in $H_1^{irreg}(U, \nabla^*)$ are

$$\xi_i = \begin{cases} \gamma_i \otimes \exp y & \text{for } i = 1, 2, \\ (\eta_{i-2} - \eta_{i-1}) \otimes \exp y & \text{for } i = 3, 4. \end{cases}$$

If we denote $\eta_{i-2} - \eta_{i-1}$ by γ_i for $i = 3, 4$, the pairing of ω_i and ξ_j is the integral

$$< \xi_j, \omega_i >= \int_{\gamma_j} \exp y \cdot \omega_i.$$

The integrations are the pairing of $H^1_{dR}(U, \nabla) \otimes \mathbb{C}$ and $H_1^{irreg}(U, \nabla)$, which produces complex numbers.

We are interested in the determinant of the matrix of the period integrals:

$$per(U, \nabla) := \det\left(< \xi_j, \omega_i >\right)_{i,j=1,2,3,4}.$$

It is not a well-defined number in $\mathbb{C}$ since we can change the basis of $H^1_{dR}(U, \nabla)$ by matrix in $\mathrm{GL}(H^1_{dR}(U, \nabla))$ and we don't have any canonical choice of a basis in $H_1^{irreg}(U, \nabla^*)$. Fixing a local solution of ∇^* yields, in this case since the monodromy is trivial, its global solution. For a chosen nonzero solution f of ∇^*, one has a $\mathbb{Q}$-vector space generated by $\gamma_i \otimes f$ for $i = 1, 2, 3, 4$ gives a $\mathbb{Q}$-structure on $H_1^{irreg}(U, \nabla^*)$. But we don't know if $c \exp y$ for a $c \in \mathbb{C}$ is more canonical than $\exp y$. Thus we have a well-defined class of numbers in $\mathbb{C}^*/k^*$ depending on the choice of the solution of ∇^*.

Throughout this article, $\exp y$ is chosen to give a $\mathbb{Q}$-structure on $H_1^{irreg}(U, \nabla^*)$.

2. Direct image connection

We keep the same convention as in the previous section.

We want to apply the product formula of Terasoma, which will be briefly reviewed in the next section. Unfortunately, it cannot be applied directly because it is done only for regular singular connections on $\mathbb{P}^1$ with some extra conditions (cf. [**27**]). So instead of calculating the determinant of the period matrix directly, we take its direct image under the projection map $\pi : X \to \mathbb{P}^1$, $(x, y) \mapsto y$ and then, we will approximate the period integral of the irregular connection $\pi_*(\nabla)$ by that of regular connections.

2.1. Splitting and the direct image connection of the ramification. We still denote by ∇ the connection on $U = \operatorname{Spec} k[x,y]/(y^2 - x(x-1)(x-\lambda)$:

$$\nabla : \mathcal{O}_U \to \mathcal{O}_U \otimes \Omega^1, \quad \text{with } \nabla(1) = dy.$$

Let π be the 2nd projection of $\mathbb{A}^2 = \{(x,y)\}$ to $\mathbb{A}^1$, $(\mathcal{O}_U, \nabla) = \pi^*((\mathcal{O}_{\mathbb{A}^1}, d + dy))$. One should be careful that here $\mathbb{A}^1$ is parametrized by y. As $\nabla = \pi^*(d + dy)$, using the Leibniz formula for tensor product of connections and the projection formula, we have

$$\begin{aligned} \pi_*(\mathcal{O}_U, \nabla) =& \pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}, d + dy) \\ =& (\mathcal{O}_{\mathbb{A}^1}, d + dy) \otimes \pi_*(\mathcal{O}_U, d). \end{aligned} \tag{38}$$

Let $f(x)$ be the polynomial $x(x-1)(x-\lambda)$. The trace map of $L := k(x,y)/(y^2 - f(x))$ in $k(y)$ has a section, which yields a splitting of $\pi_*\mathcal{O}_U$.

$$0 \to \ker(\mathrm{Tr}_{L/k(y)}) \to \pi_*\mathcal{O}_U \overset{\mathrm{Tr}_{L/k(y)}}{\longrightarrow} \mathcal{O}_{\mathbb{A}^1} \to 0.$$

$\pi_*\mathcal{O}_U$ is generated by $(1, x, x^2)$ over $\mathcal{O}_{\mathbb{A}^1}$. With respect to the basis $(1, x, x^2)$, one can calculate the $\ker(\mathrm{Tr})$ as well as the trace map:

$$\mathrm{Tr}(1) = 3, \tag{39}$$

$$\mathrm{Tr}(x) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & y^2 \\ 1 & 0 & -\lambda \\ 0 & 1 & \lambda + 1 \end{pmatrix} = \lambda + 1, \tag{40}$$

and

$$\mathrm{Tr}(x^2) = \mathrm{Tr}\begin{pmatrix} 0 & y^2 & (\lambda+1)y^2 \\ 0 & -\lambda & y^2 - \lambda(\lambda+1) \\ 1 & \lambda + 1 & (\lambda+1)^2 - 1 \end{pmatrix} = \lambda^2 + 1. \tag{41}$$

Hence for a, b, c in $k(y)$,

$$\mathrm{Tr}(a + bx + cx^2) = 3a + (\lambda + 1)b + (\lambda^2 + 1)c, \tag{42}$$

and this is in $\ker(\mathrm{Tr})$ if and only if

$$a = -\frac{\lambda + 1}{3}b - \frac{\lambda^2 + 1}{3}c.$$

Therefore $\ker(\mathrm{Tr})$ is generated by the followings:

$$v_1 := -\frac{\lambda + 1}{3} + x, v_2 := -\frac{\lambda^2 + 1}{3} + x^2.$$

Since d and Tr commute, the splitting is stable under d. Thus,

$$\pi_*(\mathcal{O}_U, d) = (\mathcal{O}_{\mathbb{A}^1}, d) \oplus (\ker(\mathrm{Tr}), \nabla_1),$$

as connections. ∇_1 has regular singularity at the roots of $(y^2-f(x_1))(y^2-f(x_2))$ where x_1, x_2 are the roots of $f'(x) = 3x^2 - 2(\lambda+1) + \lambda$.

$f'(x)$ has two distinct roots $x_1 \neq x_2$ if and only if

$$\lambda^2 - \lambda + 1 \neq 0.$$

Suppose $\lambda^2 - \lambda + 1 \neq 0$. Then ∇_1 has 4 regular singularities $\pm\sqrt{f(x_1)}, \pm\sqrt{f(x_2)}$. The connection matrix of ∇_1 in the basis v_1, v_2 is obtained from the following calculation:

(43)
$$\begin{aligned}
\nabla_1(v_1) =& d(v_1) = dx \\
=& \frac{2y}{3x^2 - 2(\lambda+1)x + \lambda} dy \\
=& \frac{(y^2 - f(x_1))(y^2 - f(x_2))}{3(x-x_1)(x-x_2)} \frac{2y}{(y^2 - f(x_1))(y^2 - f(x_2))} dy \\
=& \frac{1}{3}(f_0(y) + f_1(y)x + f_2(y)x^2) \frac{2y}{(y^2 - f(x_1))(y^2 - f(x_2))} dy,
\end{aligned}$$

where f_0, f_1, f_2 are polynomials in y.

Let N_i denote the i-th Newton polynomial of x_1, x_2 (i.e. $x_1^i + x_2^i$) and M denote x_1x_2. Then

(44)
$$
\begin{aligned}
&\frac{(y^2 - f(x_1))(y^2 - f(x_2))}{(x - x_1)(x - x_2)} \\
=&(x^2 + (x_1 - (\lambda + 1))x + x_1^2 - (\lambda + 1)x_1 + \lambda) \\
&\cdot (x^2 + (x_2 - (\lambda + 1))x + x_2^2 - (\lambda + 1)x_2 + \lambda) \\
=&x^4 + (x_1 + x_2 - 2(\lambda + 1))x^3 \\
&+ (\lambda^2 - 2(x_1 + x_2 - 2)\lambda + x_1^2 + x_2^2 + x_1x_2 - 2x_1x_2 + 1)x^2 \\
&+ ((x_1 + x_2 - 2)\lambda^2 - (x_1^2 + x_2^2 + 2x_1x_2 + 3(x_1 + x_2))\lambda \\
&+ x_1x_2(x_1 + x_2 - (x_1^2 + x_2^2) - 2x_1x_2 + x_1 + x_2 \\
&+ ((x_1x_2 - x_1 - x_2 + 1)\lambda^2 + \\
&(-x_1x_2(x_1 + x_2) + x_1^2 + x_2^2 + 2x_1x_2 - x_1 - x_2)\lambda + \\
&\quad x_1^2x_2^2 - x_1x_2(x_1 + x_2) + x_1x_2 \\
=&((-N_1 + 1)\lambda - N_1 + N_2 + M)x^2 \\
&+ ((N_1 - 1)\lambda^2 + (2N_1 - N_2 - 2M - 1)\lambda + (M + 1)N_1 - N_2 - 2M + y^2)x \\
&+ (-N_1 + M + 1)\lambda^2 + ((-M - 1)N_1 + N_2 + 2M - y^2)\lambda + (-M + y^2)N_1 \\
&+ (M^2 + M - y^2)
\end{aligned}
$$

Since,

$$N_1 = \frac{2(\lambda + 1)}{3}, \tag{45}$$

$$M = \frac{\lambda}{3}, \tag{46}$$

and

$$N_2 = \frac{2(2\lambda^2 + \lambda + 2)}{9}, \tag{47}$$

we obtain

$$f_0(y) = -\frac{1}{9}\lambda(\lambda - 1)^2 - \frac{\lambda + 1}{3}y^2, \tag{48}$$

$$f_1(y) = \frac{1}{9}(\lambda + 1)(2\lambda^2 - 3\lambda + 2) + y^2 \tag{49}$$

and

$$f_2(y) = -\frac{2}{9}(\lambda^2 - \lambda + 1). \tag{50}$$

For simplicity, we will write

$$\omega = \frac{2y}{3(y^2 - f(x_1))(y^2 - f(x_2))} dy \in \Gamma(\mathbb{A}^1, \Omega(\log D)), \tag{51}$$

for $D = \{\pm\sqrt{f(x_1)}, \pm\sqrt{f(x_2)}\}$

Then $\nabla_1(v_1) = (f_1 v_1 + f_2 v_2) \cdot \omega$.

Using the Leibniz rule,

$$\begin{aligned} \nabla_1(v_2) &= d(-\frac{\lambda^2+1}{3} + x^2) \\ &= dx^2 \\ &= 2x(f_0 + f_1 x + f_2 x^2)\omega \\ &= (2(f_0 - f_2\lambda)v_1 + 2(f_1 + (\lambda+1)f_2)v_2) \cdot \omega. \end{aligned} \tag{52}$$

Hence the connection matrix of $\ker \mathrm{Tr}$ is

$$\begin{aligned} &\begin{pmatrix} f_1 & 2(f_0 - \lambda f_2) \\ f_2 & 2(f_1 + (\lambda+1)f_2) \end{pmatrix} \cdot \omega \\ =&\frac{2}{3}\frac{ydy}{(y^2 - f(x_1))(y^2 - f(x_2))} \\ &\cdot \begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2 - 3\lambda + 2) + y^2 & \frac{2}{9}\lambda(\lambda+1) - \frac{2}{3}(\lambda+1)y^2 \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda+1) + 2y^2 \end{pmatrix} dy. \end{aligned} \tag{53}$$

with respect to v_1, v_2.

The above connection matrix has only simple poles at $y = \pm\sqrt{f(x_1)}$, $\pm\sqrt{f(x_2)}$.

Now we will calculate the same for the case $\lambda^2 - \lambda + 1 = 0$. In this case, $x_1 = x_2$ and ∇_1 has only two regular singularities $\pm\sqrt{f(x_1)}$. Via a similar calculation, we obtain its connection matrix. Note that $f(x) - f(x_1)$ has a triple root $\frac{\lambda+1}{3}$. From the coefficient of $f(x) - f(x_1)$, we have

$$x_1 = \frac{\lambda+1}{3}. \tag{54}$$

Then

$$
\begin{aligned}
\nabla_1(v_1) =& dx = \frac{2y}{3x^2 - 2(\lambda+1)x + \lambda} dy \\
=& \frac{y^2 - f(x_1)}{3(x-x_1)^2} \frac{2y}{y^2 - f(x_1)} dy \\
=& \frac{x - x_1}{3} \frac{2y}{y^2 - f(x_1)} dy \\
=& (x - \frac{\lambda+1}{3}) \frac{2y}{3(y^2 - f(x_1))} dy \\
=& \frac{2ydy}{3(y^2 - f(x_1))} v_1.
\end{aligned} \tag{55}
$$

If we denote $\dfrac{2ydy}{3(y^2 - f(x_1))}$ by ω', we have

$$\nabla_1(v_1) = \omega' \otimes v_1. \tag{56}$$

Similarly,

$$
\begin{aligned}
\nabla(v_2) =& 2xdx \\
=& 2(x^2 - \frac{\lambda+1}{3}x) \frac{2y}{3(y^2 - f(x_1)} dy \\
=& \omega' \otimes (2v_2 - \frac{2(\lambda+1)}{3} v_1)
\end{aligned} \tag{57}
$$

Thus in this case, the connection matrix is

$$\begin{pmatrix} 1 & -\frac{2}{3}(\lambda+1) \\ 0 & 2 \end{pmatrix} \omega' = \begin{pmatrix} 1 & -\frac{2}{3}(\lambda+1) \\ 0 & 2 \end{pmatrix} \frac{2ydy}{3(y^2 - f(x_1))}. \tag{58}$$

2.2. Residues. For the generic λ (i.e. $\lambda^2 - \lambda + 1 \neq 0$), the rank 2 part of $\pi_* \nabla$ is

$$d + \begin{pmatrix} dy & 0 \\ 0 & dy \end{pmatrix} + \begin{pmatrix} f_1 & 2(f_0 - f_2\lambda) \\ f_2 & 2(f_1 + f_2(\lambda+1)) \end{pmatrix} \omega,$$

with f_0, f_1, f_2 defined in (48), (49) and (50).

We will denote the rank 2 connection by $(\mathcal{V}, \nabla_{\mathcal{V}})$ for a while. This is the tensor product of $(\mathcal{O}_{\mathbb{A}^1}, d + dy)$ and $(\ker \mathrm{Tr}, \nabla_1)$.

$$(\mathcal{V}, \nabla_{\mathcal{V}}) = (\mathcal{O}_{\mathbb{A}^1}, d + dy) \otimes (\ker \mathrm{Tr}, d + A),$$

for

(59)

$$A = \begin{pmatrix} f_1 & 2(f_0 - \lambda f_2) \\ f_2 & 2(f_1 + (\lambda+1) f_2) \end{pmatrix} \omega$$
$$= \begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2 - 3\lambda + 2) + y^2 & \frac{2}{9}\lambda(\lambda^2+1) - \frac{2}{3}(\lambda+1)y^2 \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda+1) + 2y^2 \end{pmatrix} \omega.$$

As tensoring $(\mathcal{O}_{\mathbb{P}^1}, d{+}dy)$ does not change the residues at $\pm\sqrt{f(x_1)}$, $\pm\sqrt{f(x_2)}$, we only have to calculate those for $(\mathcal{V}, \nabla_{\mathcal{V}})$ in order to check if Terasoma's formula is applicable.

Since x_1, x_2 are the roots of $f'(x) = 3x^2 - (\lambda+1)x + \lambda$,

$$f(x_i) = -\frac{2}{9}(\lambda^2 - \lambda + 1)x_i + \frac{1}{9}\lambda(\lambda+1), \quad \text{for } i = 1, 2 \tag{60}$$

and the connection matrix of $(\mathcal{V}, \nabla_{\mathcal{V}})$ has at most simple poles at $\sqrt{f(x_1)}$, the residue is

(61)

$$\lim_{y \to \sqrt{f(x_i)}} (y - \sqrt{f(x_i)}) A =$$
$$\begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2 - 3\lambda + 2) + f(x_i) & \frac{2}{9}\lambda(\lambda^2+1) - \frac{2}{3}(\lambda+1)f(x_i) \\ -\frac{2}{9}(\lambda^2 - \lambda + 1) & -\frac{2}{9}\lambda(\lambda+1) + 2f(x_i) \end{pmatrix} \mathrm{Res}_{\sqrt{f(x_i)}}(\omega).$$

This is the product of

$$\mathrm{Res}_{\sqrt{f(x_i)}}\, \omega = \lim_{y \to \sqrt{f(x_i)}} (y - \sqrt{f(x_i)}) < \omega, \frac{d}{dy} >$$
$$= \frac{1}{3(f(x_i) - f(x_j))}$$

and

$$A|_{y=\sqrt{f(x_i)}} = \frac{2(\lambda^2 - \lambda + 1)}{9} \begin{pmatrix} \lambda + 1 - x_i & \frac{2}{3}((\lambda+1)x_i + \lambda) \\ -1 & -2x_i \end{pmatrix}.$$

Whereas,

$$f(x_i) - f(x_j) = -\frac{2}{9}(x_i - x_j)(\lambda^2 - \lambda + 1).$$

The trace and the determinant of the above are

$$\mathrm{Tr}\, A|_{y=\sqrt{f(x_i)}} = \frac{2(\lambda^2 - \lambda + 1)}{9}(\lambda + 1 - 3x_i),$$

$$\begin{aligned}
&\det A\big|_{y=\sqrt{f(x_i)}} \\
=&\frac{4(\lambda^2-\lambda+1)^2}{81}(2x_i^2-2(\lambda+1)x_i+\frac{2}{3}(\lambda+1)x_i+\frac{2}{3}\lambda) \\
=&\frac{4(\lambda^2-\lambda+1)^2}{81} \\
&\cdot(2(\frac{2}{3}(\lambda+1)x_i-\frac{\lambda}{3})-2(\lambda+1)x_i+\frac{2}{3}(\lambda+1)x_i+\frac{2}{3}\lambda) \\
=&0.
\end{aligned}$$

Hence the characteristic polynomial of $\mathrm{Res}_{\sqrt{f(x_i)}}(\nabla_{\mathcal{V}})$ is

$$P(\mathrm{Res}_{\sqrt{f(x_i)}}(\nabla_{\mathcal{V}}), X) = X^2 + \frac{(3x_i-(\lambda+1))}{3(x_i-x_j)}X.$$

One fixes a branch of the square root :

$$x_i = \frac{\lambda+1+\sqrt{\lambda^2-\lambda+1}}{3},$$

then automatically,

$$x_j = \frac{\lambda+1-\sqrt{\lambda^2-\lambda+1}}{3}.$$

Then one has the characteristic polynomial

$$X^2+\frac{1}{2}X.$$

Therefore, the residue at each ramification point of π has eigenvalue $0, -\frac{1}{2}$. This means Terasoma's result is not directly applicable in this case.

When $\lambda^2-\lambda+1=0$, the rank 2 part of $\pi_*(\nabla)$ is

$$d + I dy + \begin{pmatrix} 1 & -\frac{2}{3}(\lambda+1) \\ 0 & 2 \end{pmatrix}\omega'. \tag{62}$$

The residue matrix at $\sqrt{f(x_1)}$ is simply

$$\begin{pmatrix} 1 & -\frac{2}{3}(\lambda+1) \\ 0 & 2 \end{pmatrix}\mathrm{Res}_{\sqrt{f(x_1)}}\omega', \tag{63}$$

for

$$\begin{aligned}
\mathrm{Res}_{y=\sqrt{f(x_1)}}\omega' &= \lim_{y\to\sqrt{f(x_1)}}\frac{2y}{3(y^2-f(x_1))}(y-\sqrt{f(x_1)}) \\
&= \frac{2\sqrt{f(x_1)}}{3\cdot(2\sqrt{f(x_1)})} = \frac{1}{3}.
\end{aligned} \tag{64}$$

Therefore there are two eigenvalues of the residue of ∇_1 at the ramification point $\sqrt{f(x_1)}$: $1/3, 2/3$. In the next section, we will see the product formula is not directly applied to approximate the period determinant of $\pi_*(\nabla)$.

3. Terasoma's work

Let us recall the main theorem in [**27**]. The theorem tells the exact value of determinant of the period matrix of a connection on $\mathbb{A}^1$ with some extra conditions.

3.1. De Rham cohomology with compact support and relative homology. Let $D = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be n distinct points in $\mathbb{A}^1$. Then a logarithmic connection with poles at $D \cup \{\infty\}$

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega_{\mathbb{P}^1}(\log(D + \infty))$$

on a trivial bundle $\mathcal{V}$ over $\mathbb{A}^1$ can be written as

$$\nabla = d + \sum_{i=1}^{n} \frac{B^{(i)}}{x - \lambda_i},$$

here $B^{(i)}$ in $\mathrm{End}(\mathbb{C}^n)$ is called the residue matrix of ∇ at λ_i and denoted by $\mathrm{Res}_{\lambda_i}(\nabla)$. The residue of at ∞ is $B^{(\infty)} = \sum_{i=1}^{n}(-B^{(i)})$.

Throughout this section, we will assume the condition below:

Condition 15. Assume that no two eigenvalues of $B^{(i)}$ (resp. $-B^{(\infty)}$) differ by an integer and that they have positive real part.

Let us recall the notion of small extension and its properties from [**23**] and [**27**].

Definition 2. If no eigenvalue of residues is an integer less than or equal to 0, ∇ is called a small extension of $\nabla|_U$ along D.

When ∇ is small at D, then the de Rham cohomology $H^1_{dR}(\mathbb{A}^1, \nabla)$ of the logarithmic de Rham complex is generated by

$$\left(\frac{1}{x - \lambda_i} - \frac{1}{x - \lambda_{i+1}}\right) dx \otimes w, \quad \text{for } i = 1, 2, \ldots, n-1$$

where w is a vector in $\mathcal{V}$. Thus if $\mathrm{rank}\, \mathcal{V} = r$, $\mathrm{rank}\, H^1_{dR}(\mathbb{A}^1, \nabla) = r(n-1)$. This cohomology is canonically isomorphic to the cohomology $H^1_{dR}(\mathbb{A}^1, j_! j^*(\nabla))$, where j is the open embedding $j : \mathbb{A}^1 - D \hookrightarrow \mathbb{A}^1$ (cf. [**23**], [**27**]).

One can define the corresponding homology theory to make a pairing with the de Rham cohomology described previously. This is the relative homology of the pair $(\mathbb{A}^1, D)$ valued in ∇^*. It is defined as the relative homology of $(U, \cup_i D_i)$ valued in the dual local system where

D_i is a small disk around λ_i, which is independent of the choice of sufficiently small D_i. It can be seen as Borel-Moore homology which has values in the local system. A cycle in this homology is the topological cycle valued in its underlying local system. So it can be written as linear sum of

$$\gamma_\alpha \otimes f_\alpha,$$

for γ_α a relative cycle of $H_i(\mathbb{A}^1, D, \mathbb{Z})$ and f_α a branch of the solution of ∇^*. We will simply denote the homology by $H_i(\mathbb{A}^1, D, \nabla^*)$. For dimension reason, $H_i(\mathbb{A}^1, D, \nabla^*)$ vanishes for $i > 1$.

Note that it depends only on the local system of ∇^*. The main theorem of [**27**] tells us that $H_1(\mathbb{A}^1, D, \nabla^*)$ and $H^1_{dR}(\mathbb{A}^1, \nabla)$ are in perfect pairing via integration.

Let us define the pairing. For a vector v in $\mathcal{V}$ and f a branch of the solution of ∇^* over γ,

$$\begin{aligned} \omega &:= \left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}}\right) dx \otimes v \quad &&\text{in } H^1_{dR}(\mathbb{A}^1, \nabla), \\ \delta &:= \gamma \otimes f &&\text{in } H_1(\mathbb{A}^1, D, \nabla^*). \end{aligned} \tag{65}$$

as

$$< \omega, \delta > = \int_\gamma \left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}}\right) < v, f > dx.$$

To state the main theorem, we need some definitions.

3.2. Tame symbol. Let ∇ be a rank 1 connection on the trivial bundle with log poles at $\lambda_1, \lambda_2, ..., \lambda_n$ as above and the residue at λ_i is a complex number b^i whose real part is positive. Then the connection can be written in the form :

$$\nabla = d + \sum_{i=1}^{n} \frac{b^i}{x-\lambda_i}.$$

The solution of its dual connection ∇^* is a multivalued function on $U := \mathbb{C} - \{\lambda_1, \lambda_2, ..., \lambda_n\}$

$$\prod_{i=1}^{n} \exp(b^i \log(x-\lambda_i)) = \prod_{i=1}^{n} (x-\lambda_i)^{b^i}.$$

Let p be a fixed point in U and γ_i is a path from p to λ_i. Fix a branch of $\log(x-\lambda_i)$ around λ_i to have real value at $\lambda_i + \epsilon$ for $\epsilon \in \mathbb{R}_{>0}$ and continue it analytically along γ_i. Denote $D(x)_{\gamma_i}$ the above solution on γ_i of ∇^* with the chosen branch of logarithms. Then the *tame symbol* *for the rank* 1 *connection* ∇ is defined as

$$(66)\qquad \begin{aligned}(\nabla,(x-\lambda_i))_{\gamma_i} :&= \lim_{x\to\lambda_i}\frac{D(x)_{\gamma_i}}{(x-\lambda_i)^{b^i}}\\ &=\prod_{j\neq i}(\lambda_i-\lambda_j)^{b^j}.\end{aligned}$$

In the same manner, for a path γ_∞ from p to ∞, we define

$$(\nabla,\frac{1}{x})_{\gamma_\infty} := \lim_{x\to\infty} D(x)_{\gamma_\infty}x^{b^\infty}.$$

for x^{b^∞} defined by $\exp(b^\infty\log x)$ for the principal branch of the logarithm.

For a higher rank connection, we define the *tame symbol* as the tame symbol for its determinant connection.

3.3. Gamma factor. Let us take the same assumption that the residue at λ_i has only eigenvalues with positive real part. The Gamma factor is a generalization of the Gamma function for the residue:

$$(67)\qquad \Gamma_x(\nabla)=\begin{cases}\det\left(\displaystyle\int_0^\infty x^{\mathrm{Res}_\lambda(\nabla)}e^{-x}\frac{dx}{x}\right) & \text{for } x \text{ in } D,\\ \det\left(\displaystyle\int_0^\infty x^{-\mathrm{Res}_\infty(\nabla)}e^{-x}\frac{dx}{x}\right) & \text{for } x=\infty.\end{cases}$$

Since the value above is invariant under constant change of basis, it is equal to the product of the values of the Gamma function at the eigenvalues with multiplicity. Specially, when ∇ is of rank 1, the Gamma factor is same as the value of the classical Gamma function at the residue.

3.4. The product formula. Now we are prepared to state the main theorem in [**27**] with tame symbols and Gamma factors. We will assume the Condition 15. imposed on ∇.

Let $\{e_i\}_{i=1,\dots,r}$ be the basis for the underlying vector bundle of ∇ and $\{e_j^*\}$ be the dual basis. Let $\delta_j := \gamma_j-\gamma_{j+1}$ and $\mathrm{Sol}(\nabla^*)(e_q^*)$ be the branch of $\mathrm{Sol}(\nabla^*)$ analytically continued along δ with the initial value e_q^* at p. Then define

$$\omega_i(e_p) := (\frac{1}{x-\lambda_i}-\frac{1}{x-\lambda_{i+1}})dx\otimes e_p,$$

and

$$\delta_j(e_q^*) := \delta_i\otimes \mathrm{Sol}(\nabla^*)(e_q^*).$$

We define a period matrix for (i,j) as

$$A_{ij} := \left(\int_{\delta_j(e_q^*)} \omega_i(e_p) \right)_{1\le p,q\le r}.$$

With the notations introduced above, the main theorem in [**27**] gives us the exact value of the above period determinant in $\mathbb{C}^*$ for the basis of $H^1_{dR}(\mathbb{A}^1,\nabla)$ and of $H_1(\mathbb{A}^1,D,\nabla^*)$ described above.

THEOREM 16. *The determinant of the period matrix is*

$$\det\left(A_{ij}\right)_{1\le i,j\le n-1} = \prod_{i=1}^{n}(\nabla, x-\lambda)_{\gamma_i}\cdot(\nabla,1/x)^{-1}_{\gamma_\infty}\prod_{i=1}^{n}\Gamma_{\lambda_i}(\nabla)\cdot\Gamma_\infty(\nabla)^{-1}.$$

in $\mathbb{C}^*$.

PROOF. See [**27**]. □

As a direct consequence, $H^1_{dR}(\mathbb{A}^1,\nabla)$ and $H_1(\mathbb{A}^1,D,\nabla^*)$ make a perfect pairing, since the values of tame symbol and the Gamma factor never vanish.

In the other hand, knowing that the differential forms

$$\omega_i(e_p) = \left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}}\right)dx\otimes e_p$$

for $i=1,\dots,n-1$, $p=1,\dots,r$ make a basis of $H^1_{dR}(\mathbb{A}^1,\nabla)$, one can take the other basis :

$$\eta_i(e_p) = \frac{x^{i-1}dx}{\prod_{k=1}^{n}(x-\lambda_k)}\otimes e_p,$$

for $i=1,\dots,n-1$, $p=1,\dots,r$.

PROPOSITION 17. η_i *defined above generate* $H^1_{dR}(\mathbb{P}^1,\nabla)$.

PROOF. Clearly, $\eta_i(e_p)$ defines a de Rham cocycle in $H^1_{dR}(\mathbb{P}^1,\nabla)$. Since the pairing

$$H^1_{dR}(\mathbb{A}^1,\nabla)\times H_1(\mathbb{A}^1,D,\nabla^*)\xrightarrow{<,>}\mathbb{C}^*$$

is perfect by Theorem 16, it suffices to show that the period matrix

$$\left(\int_{\delta_j(e_q^*)}\eta_i(e_p)\right)_{\substack{i,j=1,\dots,n-1,\\ p,q=1,\dots,r}}$$

has nonzero determinant.

For simplicity, we will check it only for $r = 1$, nevertheless the proof for higher rank cases is essentially the same. Since $p = q = 1$, we write

$$\omega_i = (\frac{1}{x - \lambda_i} - \frac{1}{x - \lambda_{i+1}})dx$$

and

$$\eta_i = \frac{x^{i-1}}{\prod_{k=1}^{n}(x - \lambda_i)}$$

without the subscripts p, q.

Let

$$\begin{aligned} \eta'_1 :=& \eta_1, \quad \text{and} \\ \eta'_i :=& \prod_{k=1}^{i-1}(x - \lambda_k)\eta_i \\ =& \frac{dx}{\prod_{k=i}^{n}(x - \lambda_k)}, \quad \text{for } i = 2, \dots, n-1. \end{aligned}$$

Then we have the relation

$$\begin{aligned} x\eta'_i =& \frac{(x - \lambda_i + \lambda_i)dx}{(x - \lambda_i)\cdots(x - \lambda_n)} \\ =& \frac{(x - \lambda_i)dx}{\prod_{k=i}^{n}(x - \lambda_k)} + \frac{\lambda_i dx}{\prod_{k=i}^{n}(x - \lambda_k)} \\ =& \eta'_{i+1} + \lambda_i \eta_i. \end{aligned}$$

Thus we have that $\eta_i = \eta'_i + \sum_{k>i} c_k \eta'_k$ for some constants c_k, so

$$\det\left(\int_{\delta_j} \eta_i\right) = \det\left(\int_{\delta_j} \eta'_i\right).$$

To simplify further the above determinant, we now need the following lemma.

LEMMA 18. *Let k be a field of characteristic 0. For n-distinct numbers $\lambda_1, \dots, \lambda_n \in k$ $(n > 2)$, the rational function in x*

$$\frac{1}{(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)}$$

is uniquely written in the form

$$c_1(\frac{1}{x - \lambda_1} - \frac{1}{x - \lambda_2}) + c_2(\frac{1}{x - \lambda_2} - \frac{1}{x - \lambda_3}) + \dots + c_{n-1}(\frac{1}{x - \lambda_{n-1}} - \frac{1}{x - \lambda_n}).$$

PROOF. One knows from elementary algebra

$$\frac{1}{(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)} = \sum_{i=1}^{n} \frac{a_i}{x - \lambda_i}$$

for some a_i in k. Note that a_i is the residue of $\frac{1}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)}$ at λ_i.

So the above can be written again as

$$
\begin{aligned}
&a_1(\frac{1}{x-\lambda_1}-\frac{1}{x-\lambda_2})+(a_1+a_2)(\frac{1}{x-\lambda_2}-\frac{1}{x-\lambda_3}) \\
&+(a_1+a_2+a_3)(\frac{1}{x-\lambda_2}-\frac{1}{x-\lambda 3})+\cdots \\
&+(a_1+a_2+\ldots+a_{n-1})(\frac{1}{x-\lambda_{n-2}}-\frac{1}{x-\lambda_{n-1}}) \\
&+(a_1+a_2+\ldots+a_{n-1}+a_n)\frac{1}{x-\lambda_n}.
\end{aligned} \tag{68}
$$

Whence the coefficient of the last term is zero because this is the sum of the whole residues of a rational function. Once a_1 is determined, the other coefficients are uniquely determined. Since the residue is uniquely defined, we have directly the uniqueness. □

In the proof of the previous lemma, one has

$$a_1 = c_1 = \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)}.$$

Thus we have

$$
\begin{aligned}
\eta_i' =& \frac{dx}{(x-\lambda_i)\ldots(x-\lambda_n)} \\
=& \frac{1}{(\lambda_i-\lambda_{i+1})(\lambda_i-\lambda_{i+2})\cdots(\lambda_i-\lambda_n)}\omega_i \\
&+\sum_{k>i} c_k\omega_k,
\end{aligned} \tag{69}
$$

for some (unimportant) constants c_k.

Seeing

$$
\begin{aligned}
\eta_{n-1}' =& \frac{dx}{(x-\lambda_{n-1})(x-\lambda_n)} \\
=& \frac{1}{\lambda_{n-1}-\lambda_n}(\frac{1}{x-\lambda_{n-1}}-\frac{1}{x-\lambda_n})dx \\
=& \frac{1}{\lambda_{n-1}-\lambda_n}\omega_{n-1},
\end{aligned} \tag{70}
$$

we know now the determinant is

$$\det\left(\int_{\delta_j}\eta_i\right) = \frac{1}{\prod_{i<j}(\lambda_i-\lambda_j)}\det\left(\int_{\delta_j}\omega_i\right). \tag{71}$$

Therefore the determinant doesn't vanish and $\{\delta_j\}$ and $\{\omega_i\}$ for $i, j = 0, 1, \ldots, n-1$ make a perfect pairing. This finishes the proof of the proposition. $\square$

REMARK 4. One can prove the previous proposition only with a bit of calculation showing η_i is linear sum of ω_i for $i = 1, \ldots, n-1$ and vice versa. But here we used the perfectness of the pairing and calculated the value, which is indispensable to approximate the period of an irregular connection.

3.5. Example. Together with Theorem 16, the previous proposition yields the main theorem in [**26**], which plays key role in [**28**]. This example is taken from [**28**]. Here we expose the determinant of a matrix whose elements are period integrals as that arising from the pairing of the de Rham cohomology and the homology for a connection. This view enables us to apply the product formula to calculate the period determinant.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be complex numbers and $s_1, s_2, \ldots, s_n$ be complex numbers with positive real part. Define the multivalued forms as

$$\alpha_i := \prod_{k=1}^{n} (x - \lambda_k)^{s_k - 1} x^{i-1} dx, \text{ for } i = 1, \ldots, n-1.$$

Take a base point $b \in \mathbb{C} - \{\lambda_1, \ldots, \lambda_n\}$ and define the path I_i as $\gamma_{b,\lambda_{i+1}} - \gamma_{b,\lambda_i}$. Fix a branch of α_i as an analytic continuation on I_i, we can evaluate the integral

$$\int_{I_j} \alpha_i.$$

Let $\nabla = d + \sum_{k=1}^{n} \frac{dx}{x - \lambda_k}$ be a connection on $\mathbb{A}^1$. Then $H^1_{dR}(\mathbb{A}^1, \nabla)$ is generated with the differential forms

$$\eta_i := \frac{x^{i-1} dx}{\prod_k (x - \lambda_k)}.$$

In the other hand, the dual local system has the multivalued solution $\prod_k (x - \lambda_k)^{s_k}$ and its homology is generated by

$$C_j := I_j \otimes \prod_k (x - \lambda_k)^{s_k}$$

for the appropriate branch of the solution.

The pairing of η_i and C_j is by definition

$$
\begin{aligned}
< C_j, \eta_i >= & \int_{I_j} \mathrm{Sol}(\nabla^*) \cdot \eta_i \\
= & \int_{I_j} \prod_{k=1}^{n} (x-\lambda_k)^{s_k - 1} x^{i-1} dx \\
= & \int_{I_j} \alpha_i .
\end{aligned}
\tag{72}
$$

Hence the period determinant is

$$
\begin{aligned}
& \det\left(\int_{I_j} \alpha_j\right) \\
= & \frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)} \det\left(\int_{I_j} \left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}}\right) \prod_k (x-\lambda_k)^{s_k} dx\right) \\
= & \left(\frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)}\right) \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \prod_k (\nabla, x-\lambda_k)_{\gamma_k} \cdot (\nabla, \frac{1}{x})_{\infty}^{-1}
\end{aligned}
\tag{73}
$$

by Theorem 16.

To get the exact value of the above, we only have to calculate the tame symbols:

$$
\begin{aligned}
(\nabla, x-\lambda_k)_{\gamma_k} = & \prod_{i \neq k} (\lambda_k - \lambda_i)^{s_k} \\
(\nabla, \frac{1}{x})_{\gamma_\infty} = & 1.
\end{aligned}
\tag{74}
$$

Hence the determinant is equal to

$$
\begin{aligned}
& \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \frac{\prod_{i=1}^{n} \prod_{j \neq i} (\lambda_j - \lambda_i)^{s_i}}{\prod_{i>j}(\lambda_j - \lambda_i)} \\
= & \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \frac{\prod_{i \neq j}(\lambda_j - \lambda_i)}{\prod_{i>j}(\lambda_j - \lambda_i) \prod_{i=1}^{n} \prod_{j \neq i} (\lambda_j - \lambda_i)^{s_i - 1}} \\
= & \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \prod_{i<j} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \left(\prod_{j \neq i} (\lambda_j - \lambda_i)\right)^{s_i - 1}.
\end{aligned}
\tag{75}
$$

Then we have the theorem :

COROLLARY 19 ([**26**], [**30**]). *With the notations above,*

$$
\det\left(\int_{I_j} \alpha_i\right) = \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n)} \prod_{i<j} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \left(\prod_{j \neq i} (\lambda_j - \lambda_i)\right)^{s_i - 1}
$$

in $\mathbb{C}^*$.

REMARK 5. In the original statement of the previous theorem in [**26**], one has some assumptions: λ_i are real numbers and s_i are positive real numbers. In view of the determinant as the period matrix arising from the pairing of the de Rham cohomology of a connection and the homology of its dual connection, we could write the determinant in terms of tame symbol and Gamma factor, which can be defined in more general situation.

REMARK 6. In higher rank case, the forms $(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1})}dx \otimes e_i$ can be changed into $\frac{x^{i-1}dx}{\prod_{i=1}^{n-1}(x-\lambda_i)} \otimes e_i$, too. In this case, the period determinant is the original one divided by Δ^r, where Δ is the Vandermonde determinant of $\lambda_1, \ldots, \lambda_n$.

4. Period integral

In this section, we evaluate the determinant of the period determinant. We will assume $\lambda^2 - \lambda + 1 \neq 0$ throughout this section. We will handle the case $\lambda^2 - \lambda + 1 = 0$ in the next section separately.

Firstly, we will calculate the period of the rank 3 connection $\pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}, d+ dy + \varpi)$, where ϖ is the one form

(76)
$$\varpi := (\frac{1}{y+\sqrt{f(x_1)}} + \frac{1}{y-\sqrt{f(x_1)}} + \frac{1}{y+\sqrt{f(x_2)}} + \frac{1}{y-\sqrt{f(x_2)}})dy.$$

Using the projection formula, we have canonically

$$\pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}, d + dy + \varpi) = (\mathcal{O}_{\mathbb{A}^1}, d + dy) \otimes (\mathcal{O}_{\mathbb{A}^1}, d + \varpi) \otimes \pi_*(\mathcal{O}_U, d).$$

It is decomposed as direct sum of rank 1 and rank 2 connections. Thus calculating the periods separately, we can multiply them to get the whole period.

The period determinant of the above connection will be compared later with that of the original one.

4.1. Small extension.

The general theory of regular singular connections is developed in [**9**]. Deligne's Theorem of existence tells us that, for a regular singular connection (E, ∇) on a nonsingular variety U there exists a logarithmic extension

$$\bar{\nabla} : \bar{E} \to \bar{E} \otimes \Omega^1(\log D)$$

on its completion $X = U \cup D$ for a normal crossing divisor D in X.

Let $j : U \hookrightarrow X$ be the open embedding.

Definition 3. If the eigenvalues of its residue at a irreducible component D_i of D is not an integer smaller than or equal to 0 (resp. bigger than 0), $\bar{\nabla}$ is called a small (resp. big) extension of ∇ at D_i. When $\bar{\nabla}$ is a small extension of ∇ at each component of D, $\bar{\nabla}$ is called a small extension of ∇.

Remark 7. Note that the above definition is the generalization of the notion *small extension* discussed in the previous section. In previous section, the notion of small extension was only defined for meromorphic connections on the trivial bundle over $\mathbb{P}^1$. Here, we have extended it for connections on an arbitrary curve (cf. [**23**]).

The de Rham complex of a small (resp. big) extension $(\bar{E}, \bar{\nabla})$ is quasi-isomorphic to $j_!(E^\nabla)$ (resp. $Rj_*(E^\nabla)$) in strong topology (cf. Lemma 1.6 in [**23**]). Thus we have

$$\mathbb{H}^i(X, (\bar{E}, \bar{\nabla})) \simeq \begin{cases} H^i_c(U, \mathcal{E}) & \text{if } \bar{\nabla} \text{ is small,} \\ H^i(U, \mathcal{E}) & \text{if } \bar{\nabla} \text{ is big.} \end{cases} \tag{77}$$

Returning to the calculation, from now on, we take $U = \operatorname{Spec} k[x,y]/(y^2 - f(x))$ for $f(x) = x(x-1)(x-\lambda)$ and assume $f(x_1) \neq f(x_2)$, which is equivalent to $\lambda^2 - \lambda + 1 \neq 0$.

The product formula of Terasoma is valid only for small extensions. The rank 2 part of $\pi_*\nabla$ is not small at the ramification points. It has the eigenvalues $0, 1/2$ of the residues at $D = \{\pm\sqrt{f(x_1)}, \pm\sqrt{f(x_2)}\}$.

Let ∇_D be the tensor of $d + dy$ and $d + \varpi$ on $\mathcal{O}_{\mathbb{A}^1}$:

$$\begin{aligned} \nabla_D =& d + dy \\ &+ \left(\frac{1}{y - \sqrt{f(x_1)}} + \frac{1}{y + \sqrt{f(x_1)}} + \frac{1}{y - \sqrt{f(x_2)}} + \frac{1}{y + \sqrt{f(x_2)}}\right) dy. \end{aligned} \tag{78}$$

Hence we will calculate the period for the connection

$$\pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}, \nabla_D),$$

As before, applying the projection formula, we obtain

$$\begin{aligned} \pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}(-D), \nabla_D) =& (\mathcal{O}(-D), \nabla_D) \otimes \pi_*(\mathcal{O}_X, d) \\ =& (\mathcal{O}(-D), \nabla_D) \otimes ((\mathcal{O}_{\mathbb{A}^1}, d) \oplus (\mathcal{V}, \nabla_{\mathcal{V}})). \end{aligned}$$

Note $\pi_*\pi^*(\mathcal{O}_{\mathbb{A}^1}(-D), \nabla_D)$ is small since the rank 1 part has only 1 as the residue and the rank 2 part has the eigenvalues $1/2, 1$ of the residue at a point in D.

Since the connection in consideration splits into the direct sum of rank 1 and rank 2 connections, the period will be given by the product of the period of the rank 1 part with that of the rank 2 part.

One is now prepared to evaluate the period by approximation. The approximation will be made with a sequence of regular singular connections to produce the period for the irregular connection. This was already applied in [**28**] without the interpretation in terms of connection, for a special type of rank 1 connection. We will imitate the process for a higher rank connection on $\mathbb{A}^1$.

4.2. Rank 1 part. The rank 1 part has been already estimated in [**28**]. We recall the Theorem 2.3.3. in op.cit. To state the theorem, we need to reorder some notations.

Let $F(x)$ be a polynomial of degree d, $\lambda_i, \dots, \lambda_n$ be distinct complex numbers and $s_1, s_2, \dots, s_n$ be positive real numbers.

For a rank 1 connection

$$\nabla = d + dF + \sum_{i=1}^{n} \frac{s_i}{x-\lambda_i}$$

on $\mathbb{A}^1$, the de Rham cohomology is generated by

$$\eta_i = \frac{x^{i-1}dx}{\prod_{k=1}^{n}(x-\lambda_k)}, \qquad \text{for } i = 1, \dots, n-1.$$

And the homology valued in the dual connection is generated by these cycles

$$I_j \otimes \prod_{k=1}^{n}(x-\lambda_k)^{s_k-1}\exp(F(x)), \text{ for } j = 1, 2, \dots, n-1, \text{ and}$$
$$J_j \otimes \prod_{k=1}^{n}(x-\lambda_k)^{s_k-1}\exp(F(x)), \text{ for } j = 1, 2, \dots, d.$$

where I_j is a path from λ_j to λ_{j+1} and J_j is a path from λ_n to ∞ along j-th rapid decay sector.

It is a theorem on the determinant of the following matrix of confluent hypergeometric functions:

$$D = \det\Big(\Big(\int_{I_j}\prod_{k=1}^{n}(x-\lambda_k)^{s_k-1}\eta_i\Big)_{\substack{i=1,\dots,n+d-1\\ j=1,\dots,n-1}}\Big(\int_{J_j}\prod_{k=1}^{n}(x-\lambda_k)^{s_k-1}\eta_i\Big)_{\substack{i=1,\dots,n+d-1\\ j=1,\dots,d}}\Big)$$

THEOREM 20 (Terasoma[28]). *The determinant D of the confluent hypergeometric functions is the number in $\mathbb{C}^*$:*

$$
\begin{aligned}
D =&(2\pi)^{(d-1)/2}\Gamma(s_1)\cdots\Gamma(s_n)(da_d)^{-s-(d-1)/2}(-1)^{ds+d(d-1)/4}\\
&\times\prod_{i=1}^{n}(\prod_{j\neq i}(\lambda_i-\lambda_j))^{s_i-1}\prod_{i<j}(\lambda_j-\lambda_i)\\
&\times\prod_{i=1}^{n}\exp(F(\lambda_i))\prod_{F'(u)=0}\exp(F(u)),
\end{aligned}
\tag{79}
$$

where $s = s_1 + \cdots + s_n$.

PROOF. See [28]. □

In our case, the connection is

$$\nabla = d+dy+(\frac{1}{x+\sqrt{f(x_1)}}+\frac{1}{x-\sqrt{f(x_1)}}+\frac{1}{x+\sqrt{f(x_2)}}+\frac{1}{x-\sqrt{f(x_2)}})dy.$$

Hence the period determinant is

$$
\begin{aligned}
D =&(\sqrt{f(x_1)}+\sqrt{f(x_2)})(2\sqrt{f(x_1)})(\sqrt{f(x_1)}-\sqrt{f(x_2)})\\
&\times(\sqrt{f(x_1)}-\sqrt{f(x_2)})(-2\sqrt{f(x_2)})(-\sqrt{f(x_2)}-\sqrt{f(x_1)})\\
=&2^2\sqrt{f(x_1)}\sqrt{f(x_2)}(f(x_1)-f(x_2))^2
\end{aligned}
\tag{80}
$$

in $\mathbb{C}$.

4.3. Rank 2 part. Recall that the rank 2 part of $\pi_*(\mathcal{O}_U, d) = (\ker \mathrm{Tr}, \nabla_1)$ has the connection

$$\nabla_1 = d + \begin{pmatrix} f_1 & 2(f_0-f_2\lambda) \\ f_2 & 2(f_1+f_2(\lambda+1)) \end{pmatrix}\omega.$$

The rank 2 part of $\pi_*\pi^*(d + dy)$ has the connection

$$
\begin{aligned}
\nabla_D\otimes\nabla_1 =&d+Idy\\
&+I\big(\frac{1}{y-\sqrt{f(x_1)}}+\frac{1}{y+\sqrt{f(x_1)}}+\frac{1}{y-\sqrt{f(x_2)}}+\frac{1}{y+\sqrt{f(x_2)}}\big)dy\\
&+\begin{pmatrix} f_1 & 2(f_0-f_2\lambda) \\ f_2 & 2(f_1+f_2(\lambda+1)) \end{pmatrix}\omega.
\end{aligned}
\tag{81}
$$

Let e_1, e_2 be a basis of $\mathcal{V}$ and e_1^*, e_2^* be its dual basis so that $< e_i, e_j^* > = \delta_{ij}$. Let γ_i be the path from 0 to $-\sqrt{f(x_1)}, \sqrt{f(x_1)}, -\sqrt{f(x_2)}, \sqrt{f(x_2)}$, respectively for $i = 1, 2, 3, 4$. Let γ_∞ be the path from p

to ∞ along the unique rapid decaying sector around ∞. I_j is the chain $\gamma_{i+1} - \gamma_i$ for $i = 1, 2, 3$. I_4 is $\gamma_\infty - \gamma_3$.

We will take the cocycles

(82)
$$\eta_{i,a} := \frac{y^{i-1}dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})} \otimes e_a,$$
for $i = 1, \ldots, 4$ and $a = 1, 2$

as a basis for $H^1_{dR}(\mathbb{A}^1, \nabla_D \otimes \nabla_1)$, and the cycles

$$C_{j,b} := I_j \otimes (\exp y \cdot \mathrm{Sol}(\nabla^*_{\mathcal{V}}))(e^*_b), \quad \text{for } j = 1, \ldots, 4, \tag{83}$$

as a basis for $H^{irreg}_1(\mathbb{A}^1, D, (\nabla_D \otimes \nabla_1)^*)$ where the notations are as defined in Section 3.

The period pairing is the integration

(84)
$$\begin{aligned}
&P_{i,j,a,b} :=< \eta_{i,a}, C_{j,b} > \\
=&< \frac{y^{i-1}dy}{(y - \sqrt{x_1})(y + \sqrt{x_1})(y - \sqrt{x_2})(y + \sqrt{x_2})} \otimes e_a, I_j \otimes \exp y \cdot \mathrm{Sol}(\nabla^*_{\mathcal{V}})(e^*_b) > \\
=&\int_{I_j} \frac{y^{i-1} \exp y < e_a, \mathrm{Sol}(\nabla^*_{\mathcal{V}})(e^*_b) > dy}{(y - \sqrt{x_1})(y + \sqrt{x_1})(y - \sqrt{x_2})(y + \sqrt{x_2})}
\end{aligned}$$
for $i = 1, 2, 3, 4$ and $a, b = 1, 2$.

Let $I^{(m)}_j$ be I_j for $j = 1, 2, 3$ and $I^{(m)}_4$ be $\gamma_{(-m)} - \gamma_4$, where $\gamma_{(-m)}$ is a path from 0 to $-m$. The above integral can be approximated by

(85)
$$\begin{aligned}
&P_{(m),i,j,a,b} \\
=&\int_{I^{(m)}_j} \frac{y^{i-1}(1 + \frac{y}{m})^m < e_a, \mathrm{Sol}(\nabla^*_{\mathcal{V}})(e^*_b) > dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})} \\
=&(\frac{1}{m})^m \int_{I_j} \frac{y^{i-1}(y + m)^{m+1} < e_a, \mathrm{Sol}(\nabla^*_{\mathcal{V}})(e^*_b) > dy}{(y - \sqrt{f(x_1)})(y + \sqrt{f(x_1)})(y - \sqrt{f(x_2)})(y + \sqrt{f(x_2)})(y + m)},
\end{aligned}$$

as m tends to ∞.

The above, in turn, appears as the period integral of the regular singular connection

(86)
$$
\begin{aligned}
\nabla^{(m)} =& d + I\frac{(m+1)dy}{y+m} \\
&+ I\left(\frac{1}{y-\sqrt{f(x_1)}} + \frac{1}{y+\sqrt{f(x_1)}} + \frac{1}{y-\sqrt{f(x_2)}} + \frac{1}{y+\sqrt{f(x_2)}}\right)dy \\
&+ \begin{pmatrix} f_1 & 2(f_0 - f_2\lambda) \\ f_2 & 2(f_1 + f_2(\lambda+1)) \end{pmatrix} \omega.
\end{aligned}
$$

Note that the form $\frac{(m+1)dy}{y+m}$ was chosen to approximate dy. Thus we have to calculate the period for a fixed m.

Now we need to calculate its tame symbols and Gamma factors.

4.3.1. *Tame symbols.* Tame symbols depend only on the determinant bundle.

(87)
$$
\begin{aligned}
\det \nabla^{(m)} =& d + \frac{2(m+1)}{y+m}dy \\
&+ \frac{5}{2}\left(\frac{1}{y-\sqrt{f(x_1)}} + \frac{1}{y+\sqrt{f(x_1)}} + \frac{1}{y-\sqrt{f(x_2)}} + \frac{1}{y+\sqrt{f(x_2)}}\right)dy
\end{aligned}
$$

The multivalued solution of its dual local system is

$$
\mathrm{Sol}(\det \nabla^*) = (y^2 - f(x_1))^{5/2}(y^2 - f(x_2))^{5/2}(y+m)^{2(m+1)}. \tag{88}
$$

It follows that

(89)
$$
\begin{aligned}
(\nabla^{(m)}, y-\sqrt{f(x_1)})_{\gamma_1} &= (2\sqrt{f(x_1)})^{5/2}(f(x_1)-f(x_2))^{5/2}(\sqrt{f(x_1)}+m)^{2(m+1)}, \\
(\nabla^{(m)}, y+\sqrt{f(x_1)})_{\gamma_2} &= (-2\sqrt{f(x_1)})^{5/2}(f(x_1)-f(x_2))^{5/2}(-\sqrt{f(x_1)}+m)^{2(m+1)}, \\
(\nabla^{(m)}, y-\sqrt{f(x_2)})_{\gamma_3} &= (2\sqrt{f(x_2)})^{5/2}(f(x_2)-f(x_1))^{5/2}(\sqrt{f(x_2)}+m)^{2(m+1)}, \\
(\nabla^{(m)}, y+\sqrt{f(x_2)})_{\gamma_4} &= (-2\sqrt{f(x_2)})^{5/2}(f(x_2)-f(x_1))^{5/2}(-\sqrt{f(x_2)}+m)^{2(m+1)}, \\
(\nabla^{(m)}, y+m)_{\gamma_{-m}} &= (m^2 - f(x_1))^{5/2}(m^2 - f(x_2)^{5/2} \quad \text{and} \\
(\nabla, \frac{1}{y})_{\gamma_\infty} &= 1.
\end{aligned}
$$

4.3.2. *Gamma factors.* Gamma factors depend on the residue and its eigenvalues.

At $\pm\sqrt{f(x_i)}$ for $i = 1, 2$, the residue has two eigenvalues $1, 3/2$. Thus the Gamma factor at $\pm\sqrt{f(x_i)}$ for $i = 1, 2$ is

$$\Gamma_{\pm\sqrt{f(x_i)}}(\nabla) = \Gamma(1)\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

At $y = -m$, since the residue has twice of $m+1$ as eigenvalue,

$$\Gamma_{-m}(\nabla) = \Gamma(m+1)^2.$$

Let $A_\infty d(1/y)$ be the connection form of $\nabla^{(m)}$. At $y = \infty$, the connection form of ∇ is

(90)

$$\begin{aligned} A_\infty d(1/y) &:= -((m+1)+5)d(1/y)/(1/y)I \\ &- \frac{2}{3}\begin{pmatrix} \frac{1}{9}(\lambda+1)(2\lambda^2-3\lambda+2)+y^2 & \frac{2}{9}\lambda(\lambda+1)-\frac{2}{3}(\lambda+1)y^2 \\ -\frac{2}{9}(\lambda^2-\lambda+1) & -\frac{2}{9}\lambda(\lambda+1)+2y^2 \end{pmatrix} \\ &\cdot \frac{y^2}{(y^2-f(x_1))(y^2-f(x_2))}\frac{d(1/y)}{(1/y)}. \end{aligned}$$

The residue at ∞ is

$$\lim_{y\to\infty} A_\infty(1/y) = -\frac{2}{3}\begin{pmatrix} (m+1)+\frac{14}{3} & -\frac{2}{3}(\lambda+1) \\ 0 & (m+1)+\frac{16}{3} \end{pmatrix}.$$

It follows that the residue has two eigenvalues $-(m+1)+\frac{14}{3}$ and $-(m+1)+\frac{16}{3}$. Hence the Gamma factor at ∞ is

$$\Gamma_\infty(\nabla) = \Gamma((m+1)+\frac{14}{3})\Gamma((m+1)+\frac{16}{3}).$$

4.3.3. *Period determinant of* $\nabla^{(m)}$. Now we have prepared everything to calculate the period of $\nabla^{(m)}$: the tame symbols and the Gamma factors.

Let

$$C^{(m)}_{j,b} := I^{(m)}_j \otimes (y+m)^{m+1}\,\mathrm{Sol}(\nabla^*_{\mathcal{V}})(e^*_b)$$

and

$$\omega^{(m)}_{1a} := (\frac{1}{y+\sqrt{x_1}} - \frac{1}{y-\sqrt{x_1}})dy \otimes e_a, \tag{91}$$

$$\omega^{(m)}_{2a} := (\frac{1}{y-\sqrt{x_1}} - \frac{1}{y+\sqrt{x_2}})dy \otimes e_a, \tag{92}$$

$$\omega^{(m)}_{3a} := (\frac{1}{y+\sqrt{x_2}} - \frac{1}{y-\sqrt{x_2}})dy \otimes e_a, \tag{93}$$

$$\omega^{(m)}_{4a} := (\frac{1}{y-\sqrt{x_1}} - \frac{1}{y+m})dy \otimes e_a. \tag{94}$$

Applying Terasoma's product formula, the period of $\nabla^{(m)}$ is

$$
\begin{aligned}
D_{(m)} :=& \det\left(< \omega^{(m)}_{i,a}, J^{(m)}_{j,b} >\right)_{\substack{1\le i\le 4, 1\le a\le 2,\\ 1\le j\le 4, 1\le b\le 2}} \\
=& 2^6\pi^2 f(x_1)^{5/2} f(x_2)^{5/2}(f(x_1)-f(x_2)) \\
& \times (m^2-f(x_1))^{2(m+1)+5/2}(m^2-f(x_2))^{2(m+1)+5/2} \\
& \times \frac{\Gamma(m+1)^2}{\Gamma((m+1)+\frac{14}{3})\Gamma((m+1)+\frac{16}{3})}.
\end{aligned}
\tag{95}
$$

The determinant of period matrix $P_{(m)}$ is the complex number converging to the determinant of the period matrix $(P_{i,j,a,b})$ defined in (84)

$$
\begin{aligned}
\det P_{(m)} =& \det\left(P_{(m),i,j,a,b}\right) \\
=& (\frac{1}{m})^{8m}\frac{D_{(m)}}{\Delta^2_{(m)}}
\end{aligned}
\tag{96}
$$

where $\Delta_{(m)}$ is the Vandermonde determinant for $\pm\sqrt{f(x_1)}, \pm\sqrt{f(x_2)}, -m$:

(97)

$$
\begin{aligned}
&\Delta_{(m)} = \\
&(\sqrt{f(x_2)}+\sqrt{f(x_1)})2\sqrt{f(x_1)}(-\sqrt{f(x_2)}+\sqrt{f(x_1)})(-m+\sqrt{f(x_1)}) \\
&\times(\sqrt{f(x_1)}-\sqrt{f(x_2)})(-2\sqrt{f(x_2)})(-m-\sqrt{f(x_2)}) \\
&\times(-\sqrt{f(x_2)}-\sqrt{f(x_1)})(-m-\sqrt{f(x_1)}) \\
&\times(-m+\sqrt{f(x_2)}) \\
&= -2^2\sqrt{f(x_1)}\sqrt{f(x_2)}(f(x_2)-f(x_1))^2(m^2-f(x_1))(m^2-f(x_2)).
\end{aligned}
$$

Thus the period is

$$
\begin{aligned}
&(\frac{1}{m})^{8m}\frac{P_m}{\Delta^2_{(m)}}\\
=&(\frac{1}{m})^{8m}2^2\pi^2 f(x_1)^{3/2}f(x_2)^{3/2}(f(x_1)-f(x_2))^{-3}\\
&\times(m^2-f(x_1))^{2(m+1)+1/2}(m^2-f(x_2))^{2(m+1)+1/2}\\
&\times\frac{\Gamma(m+1)^2}{\Gamma((m+1)+\frac{14}{3})\Gamma((m+1)+\frac{16}{3})}\\
=&\frac{2^2\pi^2 f(x_1)^{3/2}f(x_2)^{3/2}}{(f(x_1)-f(x_2))^3}\\
&\times(\frac{1}{m})^{8m}(m^2-f(x_1))^{2m}(m^2-f(x_2))^{2m}\\
&\times m^{-10}(m^2-f(x_1))^{5/2}(m^2-f(x_2))^{5/2}\\
&\times m^{10}\frac{\Gamma(m+1)^2}{\Gamma((m+1)+\frac{14}{3})\Gamma((m+1)+\frac{16}{3})}.
\end{aligned}\tag{98}
$$

As m goes to ∞,

$$(\frac{1}{m})^{8m}(m^2-f(x_1))^{2m}(m^2-f(x_2))^{2m}\to 1\tag{99}$$

$$m^{-10}(m^2-f(x_1))^{5/2}(m^2-f(x_2))^{5/2}\to 1\tag{100}$$

$$m^{10}\frac{\Gamma(m+1)^2}{\Gamma((m+1)+\frac{14}{3})\Gamma((m+1)+\frac{16}{3})}\to 1.\tag{101}$$

Therefore the period determinant for $(\mathcal{V},\nabla_D\otimes\nabla_{\mathcal{V}})$ is

$$\frac{2^2\pi^2 f(x_1)^{3/2}f(x_2)^{3/2}}{(f(x_1)-f(x_2))^3}.$$

Now we have the period determinant of $\pi_*\pi^*(\nabla_D)$:

$$
\begin{aligned}
&(\text{Period of }(\mathcal{O}_{\mathbb{A}^1},\nabla_D))\times(\text{Period of }(\mathcal{V},\nabla_D\otimes\nabla_{\mathcal{V}}))\\
=&\frac{2^4\pi^2 f(x_1)^2 f(x_2)^2}{f(x_1)-f(x_2)}.
\end{aligned}\tag{102}
$$

4.4. Comparison. Recall U is the affine Legendre elliptic curve (i.e. the plane curve defined by $y^2=x(x-1)(x-\lambda)$) and $\nabla=d+dy$ is a connection on $\mathcal{O}_U$ as in the beginning. In this part, we will compare the period obtained previously and the period of ∇ on U. $D=\{\pm\sqrt{f(x_1)},\pm\sqrt{f(x_2)}\}$ is the divisor in $\mathbb{P}^1$ and $\Sigma=\pi^{-1}D$. Let $\pi_*(\nabla)_D$ be the rank 3 connection on $\mathbb{A}^1$

$$\pi_*(\nabla)_D:=d_D\otimes\pi_*\nabla,$$

whose period we have just calculated, where

(103)
$$d_D := d + \left(\frac{1}{y-\sqrt{f(x_1)}} + \frac{1}{y+\sqrt{f(x_1)}} + \frac{1}{y-\sqrt{f(x_2)}} + \frac{1}{y+\sqrt{f(x_2}}\right)dy.$$

(i.e. the standard exterior differentiation of $\mathcal{O}(-D)$ valued in $\mathcal{O}$.)

In the same manner, we define $\nabla_\Sigma := d_\Sigma \otimes \nabla$.

Then we have the canonical isomorphism of de Rham cohomologies

$$\pi^* : H^1_{dR}(\mathbb{A}^1, \nabla_D) \to H^1_{dR}(U, \nabla_\Sigma).$$

Let $\omega = \pi^*\eta$ be a de Rham form in $H^1_{dR}(U, \nabla_\Sigma)$ for a form η in $H^1_{dR}(\mathbb{A}^1, \nabla_D)$ and γ be a cycle in the homology of U with values in the dual local system. Then, as the pairing is functorial under change of variables,

$$\begin{aligned} \int_\gamma \omega &= \int_\gamma \pi^*\eta \\ &= \int_{\pi_*\gamma} \eta. \end{aligned} \tag{104}$$

Hence the period (102) of $\pi_*(\nabla)_D$ on $\mathbb{A}^1$ is the same as that of ∇_Σ on U.

The following short exact sequence of de Rham complexes

$$\begin{array}{ccccc} \mathcal{O}_U(-\Sigma) & \longrightarrow & \mathcal{O}_U & \longrightarrow & \mathcal{O}_\Sigma \\ \downarrow{\scriptstyle \nabla_\Sigma} & & \downarrow{\scriptstyle \nabla} & & \downarrow \\ \Omega_U(-\Sigma)(\log \Sigma) & = & \Omega_U & \longrightarrow & 0 \end{array} \tag{105}$$

yields

(106)
$$H^0_{dR}(U, \nabla) \to H^0(U, \mathcal{O}_\Sigma) \xrightarrow{\nabla} H^1_{dR}(U, \nabla_\Sigma) \to H^1_{dR}(U, \nabla) \to H^1(U, \mathcal{O}_\Sigma)$$

Whereas $H^0_{dR}(U, \nabla) = 0$, for ∇ has no (single-valued) solution on U and $H^1(U, \mathcal{O}_\Sigma) = 0$ for dimensional reason. Thus $H^1_{dR}(U, \nabla_\Sigma)$ is an extension of $H^1_{dR}(U, \nabla)$ by $H^0(U, \mathcal{O}_\Sigma)$.

In the other hand, the relative homology $H_1^{irreg}(U, \Sigma, \nabla^*)$ is isomorphic to $H_1^{irreg}(\mathbb{A}^1, D, \pi_*(\nabla^*))$. Recall the homology for a irregular connection is defined as the homology of the complex $\mathcal{C}_*^{irreg}(U, \nabla^*)$ generated by elements $c \otimes \epsilon$ with $c : \Delta^n \to X$ and $\epsilon \in \mathcal{L}_{c(b)}$ for b the barycenter of Δ^n, $\mathcal{L}$ the local system of ∇^* and ϵ decays rapidly near ∞.

The relative cohomology appears in the short exact sequence of the singular complexes valued in ∇^*:

$$0 \to \mathcal{C}_*^{irreg}(\Sigma, \nabla^*) \to \mathcal{C}_*^{irreg}(U, \nabla^*) \to \mathcal{C}_*^{irreg}(U, \Sigma, \nabla^*) \to 0,$$

which yields also the long exact sequence of homologies:

$$\begin{aligned} ... \to H_1^{irreg}(\Sigma, \nabla^*) \to H_1^{irreg}(U, \nabla^*) \to H_1^{irreg}(U, \Sigma, \nabla^*) \\ \to H_0^{irreg}(\Sigma, \nabla^*) \to H_0^{irreg}(U, \nabla^*) \to ... \end{aligned} \tag{107}$$

The exact definition of the singular complex for an irregular connection can be found in [**4**].

In the above sequence, $H_1^{irreg}(\Sigma, \nabla^*)$ and $H_0^{irreg}(U, \nabla^*)$ vanish, respectively by dimension reason and by the duality of the de Rham cohomology and the homology for irregular connections.

THEOREM 21. *Let $\gamma \otimes \mathrm{Sol}(\nabla^*)$ be a cycle in $H_1^{irreg}(U, \nabla^*)$, thus it is also a cycle in $H_1^{irreg}(U, \Sigma, \nabla^*)$. Suppose $\omega = \nabla f$ for a function f in $H^0(U, \mathcal{O}_\Sigma)$. Then the pairing of $\gamma \otimes \mathrm{Sol}(\nabla^*)$ with ∇f is* 0.

PROOF. The pairing is given as the integration

$$< \gamma \otimes \mathrm{Sol}(\nabla^*), \nabla >= \int_\gamma < \mathrm{Sol}(\nabla^*), \nabla f > .$$

This is equal to

$$\begin{aligned} &\int_\gamma (d < \mathrm{Sol}(\nabla^*), f > - < \nabla^*(\mathrm{Sol}(\nabla^*)), f >) \\ =&\int_\gamma d < \mathrm{Sol}(\nabla^*), f > \\ =&\int_{\partial\gamma} < \mathrm{Sol}(\nabla^*), f > \end{aligned} \tag{108}$$

by Stokes' theorem.

If γ is a closed cycle, its boundary $\partial\gamma$ is 0. Otherwise, we can assume without loss of generality,

$$\gamma : [0, 1] \to U \cup \{\infty\}$$

such that $\gamma(0) = \gamma(1) = \infty$, moreover, for some small $\epsilon > 0$, $\gamma(0, \epsilon)$, $\gamma(1 - \epsilon, 1)$ are inside the rapid decay sector around ∞.

Thus

$$\int_\gamma \nabla f = \lim_{\epsilon \to 0} [< \mathrm{Sol}(\nabla^*), f >]_{\gamma(\epsilon)}^{\gamma(1-\epsilon)} = 0,$$

since $\mathrm{Sol}(\nabla^*)$ decays rapidly along the sector where γ goes along. □

Let us denote by $per(\Sigma, \nabla)$ the period determinant of the pairing of $H^0(U, \mathcal{O}_\Sigma)$ and $H_0^{irreg}(\Sigma, \nabla^*)$ for suitable choice of basis. If we take a basis of $H^1_{dR}(U, \Sigma, \nabla)$ containing $\{\nabla(f_\alpha)\}_\alpha$ where $\{f_\alpha\}_\alpha$ is a basis of $H^0(\Sigma, \nabla)$, it follows

Corollary 22. *The determinant of period of* ∇_Σ *is*

$$per(U, \Sigma, \nabla_\Sigma) = per(U, \nabla) \times per(\Sigma, \nabla).$$

in $\mathbb{C}^*/k^*$.

Σ is given by the ideal

$$I = (y^2 - f(x_1))(y^2 - f(x_2))$$

in $R := k[x, y]/(y^2 - x(x-1)(x-\lambda))$.

Hence,

$$H^0(U, \mathcal{O}_\Sigma) = R/I.$$

This is the k-vector space of dimension 8 generated by

$$1, y, y^2, y^3, x, xy, xy^2, xy^3.$$

And the homology $H_0^{irreg}(\Sigma, \nabla^*)$ is generated by

$$p \otimes \exp y,$$

for $p_i \in \Sigma$.

Let x_3 (resp. x_4) be the root of $f(x) - f(x_1)$ (resp. of $f(x) - f(x_2)$), respectively, such that $x_3 \neq x_1$ (resp. $x_4 \neq x_2$). A point p in Σ can be written as $(x_i, \pm\sqrt{f(x_i)})$ and the pairing is given as

$$< p \otimes \exp y, x^a y^b >= x_i^a (\pm\sqrt{f(x_i)})^b \cdot \exp(\pm\sqrt{f(x_i)}),$$

for $a = 0, 1$ and $b = 1, 2, 3, 4$.

Let M_i be the following matrix:

$$M_i := \begin{pmatrix} 1 & \sqrt{f(x_i)} & f(x_i) & f(x_i)\sqrt{f(x_i)} \\ 1 & -\sqrt{f(x_i)} & f(x_i) & -f(x_i)\sqrt{f(x_i)} \end{pmatrix}$$

for $i = 1, 2$.

Then the period matrix in consideration is

$$Q := L \begin{pmatrix} M_1 & x_1 M_1 \\ M_2 & x_2 M_2 \\ M_1 & x_3 M_1 \\ M_2 & x_4 M_2 \end{pmatrix},$$

where $L = (l_{ij})$ is a $(8,8)$-diagonal matrix with entries

$$l_{ii} = \begin{cases} e^{f(x_1)} & \text{for } i \equiv 1 \pmod 4 \\ e^{-f(x_1)} & \text{for } i \equiv 2 \pmod 4 \\ e^{f(x_2)} & \text{for } i \equiv 3 \pmod 4 \\ e^{-f(x_2)} & \text{for } i \equiv 0 \pmod 4 \end{cases}$$

It follows that its determinant is

$$\begin{aligned} \det Q = \det \begin{pmatrix} M_1 & x_1 M_1 \\ M_2 & x_2 M_2 \\ M_1 & x_3 M_1 \\ M_2 & x_4 M_2 \end{pmatrix} \\ = \begin{pmatrix} M_1 & x_1 M_1 \\ M_2 & x_2 M_2 \\ 0 & (x_3 - x_1) M_1 \\ 0 & (x_4 - x_2) M_2 \end{pmatrix} \\ = (x_3 - x_1)^2 (x_4 - x_2)^2 \Delta_\Sigma^2, \end{aligned} \tag{109}$$

where Δ_Σ is the Vandermonde determinant for $\sqrt{f(x_1)}, -\sqrt{f(x_1)}, \sqrt{f(x_2)}, -\sqrt{f(x_2)}$. This is

$$\begin{aligned} \Delta_\Sigma = & -2\sqrt{f(x_1)}(\sqrt{f(x_2)} - \sqrt{f(x_1)})(-\sqrt{f(x_2)} - \sqrt{f(x_1)}) \\ & \times (\sqrt{f(x_2)} + \sqrt{f(x_1)})(-\sqrt{f(x_2)} + \sqrt{f(x_1)}) \\ & \times (-2\sqrt{f(x_2)}) \\ = & 2^2 \sqrt{f(x_1)}\sqrt{f(x_2)}(f(x_2) - f(x_1))^2. \end{aligned} \tag{110}$$

x_1, x_3 are the two roots of

$$\frac{f(x) - f(x_1)}{x - x_1} = x^2 + (x_1 - (\lambda+1))x + x_1^2 - (\lambda+1)x_1 + \lambda.$$

So,

$$(x_1 - x_3)^2 = \frac{-3x_1^2 + 2(\lambda+1)x_1 + (\lambda+1)^2}{4}.$$

And by the same way, we obtain

$$(x_2 - x_4)^2 = \frac{-3x_2^2 + 2(\lambda+1)x_2 + (\lambda+1)^2}{4}.$$

and again

$$(x_1 - x_3)^2 (x_2 - x_4)^2 = 2^{-4}(\lambda^2 - \lambda + 1)^2. \tag{111}$$

Altogether,

$$\begin{aligned}\det Q =&(x_3 - x_1)^2(x_4 - x_2)^2\Delta_\Sigma^2 \\ =&(\lambda^2 - \lambda + 1)^2 f(x_1)f(x_2)(f(x_2) - f(x_1))^4.\end{aligned} \tag{112}$$

At last we conclude that the period of ∇ over U is the value

$$\frac{2^4\pi^2 f(x_1)f(x_2)}{(\lambda^2 - \lambda + 1)^2(f(x_1) - f(x_2))^5}.$$

We calculate now the values $f(x_1)f(x_2)$ and $f(x_2) - f(x_1)$:

(113)

$$f(x_1)f(x_2) = -\frac{1}{3^3}\lambda^2(\lambda - 1)^2,$$

and

$$\begin{aligned}(f(x_1) - f(x_2))^2 =&(f(x_1) + f(x_2))^2 - 4f(x_1)f(x_2) \\ =&\frac{2^4}{3^6}(\lambda^2 - \lambda + 1)^3.\end{aligned}$$

It follows that

$$f(x_1) - f(x_2) = \frac{2^2}{3^3}(\lambda^2 - \lambda + 1)\sqrt{\lambda^2 - \lambda + 1},$$

for a suitable choice of the branch of the square root.

Finally, we get the main result of this chapter:

THEOREM 23. *The period determinant of $\nabla = d + dy$ over the affine Legendre elliptic curve $y^2 = x(x-1)(x-\lambda)$ defined for $\lambda \neq 0, 1, \frac{1 \pm \sqrt{3}i}{2}$ is*

$$-2^{-6}3^{12}\pi^2\frac{\lambda^2(\lambda - 1)^2}{(\lambda^2 - \lambda + 1)^9\sqrt{\lambda^2 - \lambda + 1}}$$

in $\mathbb{C}^*/k^*$.

5. Exceptional case: $\lambda^2 - \lambda + 1 = 0$

Finally, we handle the case $\lambda^2 - \lambda + 1 = 0$. Recall $x_1 = \frac{\lambda+1}{3}$ is the double root of $f'(x)$ as well as the triple root of $f(x) - f(x_1)$.

As we have seen before,

$$\pi_*(\nabla) \simeq (\mathcal{O}_{\mathbb{A}^1}, d + dy) \oplus (\ker \mathrm{Tr}, \nabla_1 \otimes (d + dy)), \tag{114}$$

where the rank 2 part has the connection

$$d + I dy + \begin{pmatrix} 1 & -\frac{2}{3}(\lambda + 1) \\ 0 & 2 \end{pmatrix}\omega'$$

as calculated in Section 3. Let us take ∇_D the above connection $\nabla_1 \otimes (d + dy)$ for short.

Since $H^1_{dR}(\mathbb{A}^1, d + dy)$ is trivial, the period of $\pi_*(\nabla)$ is equal to that of $(\ker \mathrm{Tr}, \nabla_D)$. $(\ker \mathrm{Tr}, \nabla_D)$ is of rank 2 and has two regular singularities at $y = \pm\sqrt{f(x_1)}$.

Let $\Sigma := (x_1, \pm\sqrt{f(x_1)})$ in U. Then we have the following short exact sequence:

$$0 \to H^0(\Sigma, \mathcal{O}_\Sigma) \xrightarrow{\nabla} H^1_{dR}(U, \nabla_\Sigma) \to H^1_{dR}(U, \nabla) \to 0, \tag{115}$$

Since $\mathrm{rank}\, H^1_{dR}(U, \nabla) = 4$ and $\mathrm{rank}\, H^0(\Sigma, \mathcal{O}_\Sigma) = 2$, the rank of $H^1_{dR}(U, \nabla_\Sigma)$ is 6. Note that π^* is the canonical isomorphism between $H^1_{dR}(U, \nabla_\Sigma)$ and $H^1_{dR}(\mathbb{A}^1, \nabla_D)$.

In the other hand the homologies make the following short exact sequence:

$$0 \to H_1^{irreg}(U, \nabla^*) \to H_1^{irreg}(U, \Sigma, \nabla^*) \xrightarrow{\delta} H_0^{irreg}(\Sigma, \nabla^*) \to 0. \tag{116}$$

Therefore the period we want to calculate satisfies

$$per(U, \nabla_\Sigma) = per(\Sigma, \nabla) \cdot per(U, \nabla). \tag{117}$$

Now $per(U, \nabla_\Sigma)$ will be approximated as in the previous section. For the approximation, we need to modify the cycles. Let γ_1, γ_2 a fixed path from 0 to $-\sqrt{f(x_1)}$ and to $\sqrt{f(x_1)}$ in $\mathbb{A}^1$, respectively. γ_m is a path from 0 to $-m$. Now define $I_1^{(m)} := \gamma_2 - \gamma_1$ and $I_2^{(m)} := \gamma_m - \gamma_2$. Note that $I_1^{(m)}$ is independent of m and $I_2^{(m)}$ goes to infinity along the rapid decay sector.

In the other hand, let $\eta_i(e_q)$ be $\dfrac{y^{i-1}}{y^2 - f(x_1)} dy \otimes e_q$ for $i = 1, 2$, which generate $H^1_{dR}(\mathbb{A}^1, \nabla_D)$. We still denote by $\mathrm{Sol}(\nabla, v)$ the solution of a connection ∇ with an initial value v at 0.

It is the determinant of the matrix whose elements are the pairings

$$< I_j \otimes \exp y \,\mathrm{Sol}(\nabla_1^*, e_p^*), \eta_i(e_q) >= \int_{I_j} < \mathrm{Sol}(\nabla_1^*, e_p^*), e_q > \exp y \cdot \eta_i \tag{118}$$

for $p, q = 1, 2$ and $i, j = 1, 2$.

Denote $\nabla_{(m)}$ by the connection $\nabla_1 \otimes (d + \frac{m+1}{m} dy)$. Then the above integral is

$$\begin{aligned} &\lim_{m\to\infty} \int_{I_j^{(m)}} \frac{1}{m^m} < \mathrm{Sol}(\nabla_1^*, e_p^*), e_q > \frac{(y+m)^m y^{i-1}}{(y^2 - f(x_1))} dy \\ &\qquad = \lim_{m\to\infty} < \eta_i^{(m)}(e_p), I_j^{(m)} \otimes \mathrm{Sol}(\nabla_{(m)}, e_q^*) > \end{aligned} \tag{119}$$

where $\eta_i^{(m)}(e_p)$ is the de Rham form $\frac{y^{i-1}dy}{(y^2-f(x_1))(y+m)} \otimes e_p$.

Let us denote its determinant by P_m.

Applying the product formula, we obtain

$$\begin{aligned}
P_m & \\
=&\frac{1}{m^{4m}\Delta_{(m)}^2}(\nabla_{(m)}, y-\sqrt{f(x_1)})_{\gamma_1}\cdot(\nabla_{(m)}, y+\sqrt{f(x_1)})_{\gamma_2} \\
&\times(\nabla_{(m)}, y+m)_{\gamma_m}\cdot(\nabla_{(m)}, \frac{1}{y})_{\infty}^{-1} \\
&\times\Gamma(\nabla_{(m)})_{-\sqrt{f(x_1)}}\cdot\Gamma(\nabla_{(m)})_{\sqrt{f(x_1)}}\cdot\Gamma(\nabla_{(m)})_{-m}\cdot\Gamma(\nabla_{(m)})_{\infty}^{-1}.
\end{aligned} \tag{120}$$

The values of the tame symbols in the above equality depend only on the determinant connection of ∇_m and the (multivalued) solution of its dual connection:

$$\begin{aligned}
\det\nabla_{(m)} =& d+\frac{dy}{y-\sqrt{f(x_1)}}+\frac{dy}{y+\sqrt{f(x_1)}}+\frac{2(m+1)dy}{y+m} \\
\mathrm{Sol}(\det\nabla_{(m)}^*) =& (y^2-f(x_1))(y+m)^{2(m+1)}.
\end{aligned} \tag{121}$$

Hence the tame symbols are

$$(\nabla_{(m)}, x-\sqrt{f(x_1)})_{\gamma_1} = 2\sqrt{f(x_1)}(\sqrt{f(x_1)}+m)^{2(m+1)}, \tag{122}$$

$$(\nabla_{(m)}, x+\sqrt{f(x_1)})_{\gamma_2} = -2\sqrt{f(x_1)}(-\sqrt{f(x_1)}+m)^{2(m+1)}, \tag{123}$$

$$(\nabla_{(m)}, x+m)_{\gamma_m} = m^2-f(x_1), \tag{124}$$

and

$$(\nabla_{(m)}, \frac{1}{x})_{\infty} = 1. \tag{125}$$

And the Gamma factors are

$$\Gamma_{\pm\sqrt{f(1)}}(\nabla_{(m)}) = \Gamma(\frac{1}{3})\Gamma(\frac{2}{3}), \tag{126}$$

$$\Gamma_{-m}(\nabla_{(m)}) = \Gamma(m+1)^2, \tag{127}$$

$$\Gamma_{\infty}(\nabla_{(m)}) = \Gamma(\frac{2}{3}+(m+1))\Gamma(\frac{4}{3}+(m+1)). \tag{128}$$

Altogether, we get the period P_m as the product of the followings:

$$\Gamma(\frac{1}{3})^2\Gamma(\frac{2}{3})^2 \tag{129}$$

$$\frac{(m^2 - f(x_1))^{2m}}{m^{4m}} \to \quad 1 \tag{130}$$

$$\frac{m^2 - f(x_1)}{m^2} \to \quad 1 \tag{131}$$

$$m^2 \frac{\Gamma(m+1)^2}{\Gamma((m+1)+\frac{2}{3})\Gamma((m+1)+\frac{4}{3})} \to \quad 1. \tag{132}$$

From the functional equation satisfied by the Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we obtain

$$\begin{aligned} \Gamma(\frac{1}{3})^2\Gamma(\frac{2}{3})^2 =& (\frac{\pi}{\sin\frac{\pi}{3}})^2 \\ =& \frac{2^2\pi^2}{3}. \end{aligned} \tag{133}$$

Therefore altogether it follows

$$\begin{aligned} per(U, \nabla_\Sigma) =& \Gamma(\frac{1}{3})^2\Gamma(\frac{2}{3})^2 \\ =& \frac{2^2\pi^2}{3}. \end{aligned} \tag{134}$$

Since $1, y$ is a basis of $\Gamma(\mathcal{O}_\Sigma)$

$$\begin{aligned} per(\Sigma, \nabla) =& \det \begin{pmatrix} 1 & -\sqrt{f(x_1)} \\ 1 & \sqrt{f(x_1)} \end{pmatrix} \\ =& 2\sqrt{f(x_1)}. \end{aligned} \tag{135}$$

From $x_1 = \frac{\lambda+1}{3}$ and $\lambda^2 - \lambda + 1 = 0$, it follows

$$f(x_1) = \frac{2\lambda - 1}{9} = \frac{\pm\sqrt{-3}}{9}.$$

Finally, we obtain the period

$$\begin{aligned} per(U, \nabla) =& \frac{per(U, \nabla_\Sigma)}{per(\Sigma, \nabla)} \\ =& \frac{2^2/3 \cdot \pi^2}{2/3(-3)^{1/4}} \\ =& \frac{2\pi^2}{(-3)^{1/4}} \end{aligned} \tag{136}$$

in $\mathbb{C}^*/k^*$.

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